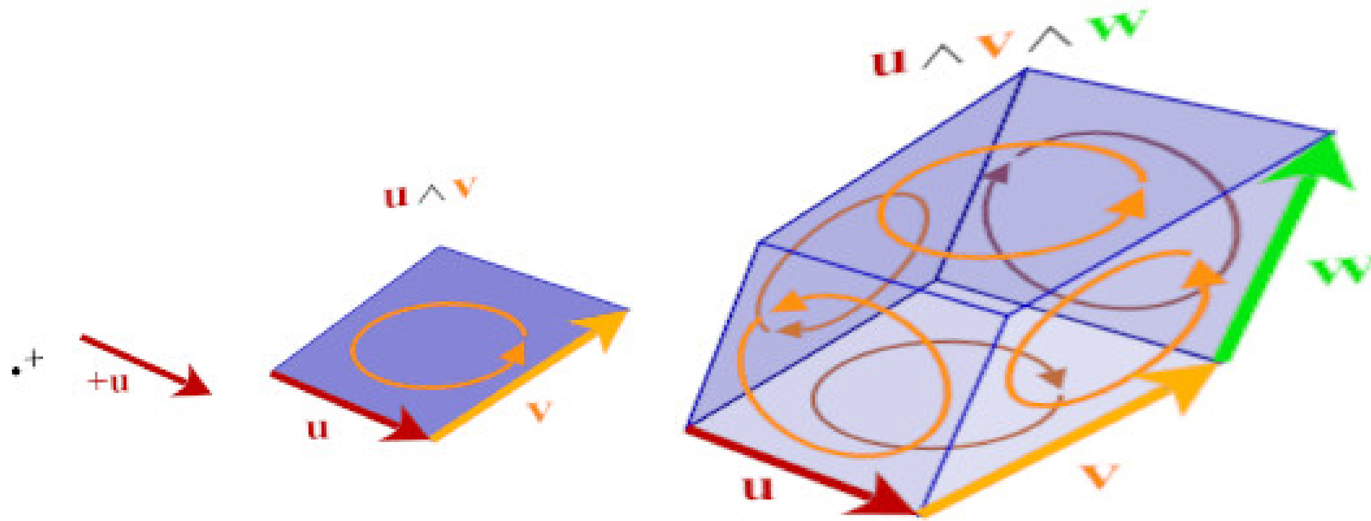
Introduction to Curves, Vector Products, and Geometric Algebra



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About Me

Pavel Loskot joined the ZJU-UIUC Institute as Associate Professor in January 2021. He received his PhD degree in Wireless Communications from the University of Alberta in Canada, and the MSc and BSc degrees in Radioelectronics and Biomedical Electronics, respectively, from the Czech Technical University of Prague. He is the Senior Member of the IEEE, Fellow of the HEA in the UK, and the Recognized Research Supervisor of the UKCGE.

In the past 25 years, he was involved in numerous industrial and academic collaborative projects in the Czech Republic, Finland, Canada, the UK, Turkey, and China. These projects concerned mainly wireless and optical telecommunication networks, but also genetic regulatory circuits, air transport services, and renewable energy systems. This experience allowed him to truly understand the interdisciplinary workings, and crossing the disciplines boundaries.

His current research focuses on statistical signal processing and importing methods from Telecommunication Engineering and Computer Science to model and analyze systems more efficiently and with greater information power.

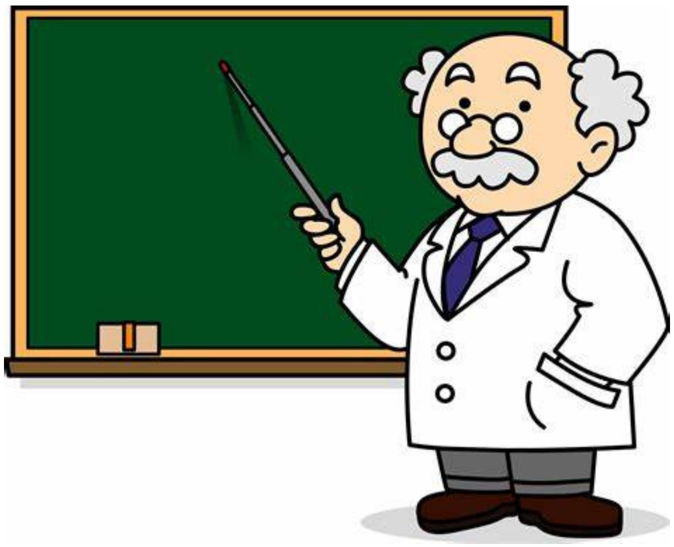
Objectives

Explore basic ideas:

- about a few chosen topics in applied mathematics
- create understanding and raise awareness about what exist
- initially (this talk), allow for simplifications and inaccuracies
- inspire applications outside mathematics
  $\rightarrow$ engineering, machine learning

Topics

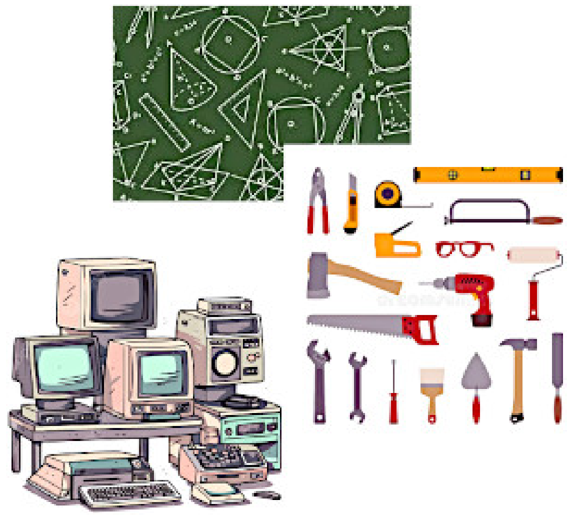
1. Mathematical objects
2. Products between these objects
3. Geometric algebra
4. Curves and splines



MOTIVATION

Mathematics

- focus on accuracy and generating fundamental knowledge
- applied mathematics now also include numerical methods (and AI/ML)
  → strong overlap with Computer Science
- widespread use of mathematical modeling
  → mathematical physics (reality problems)

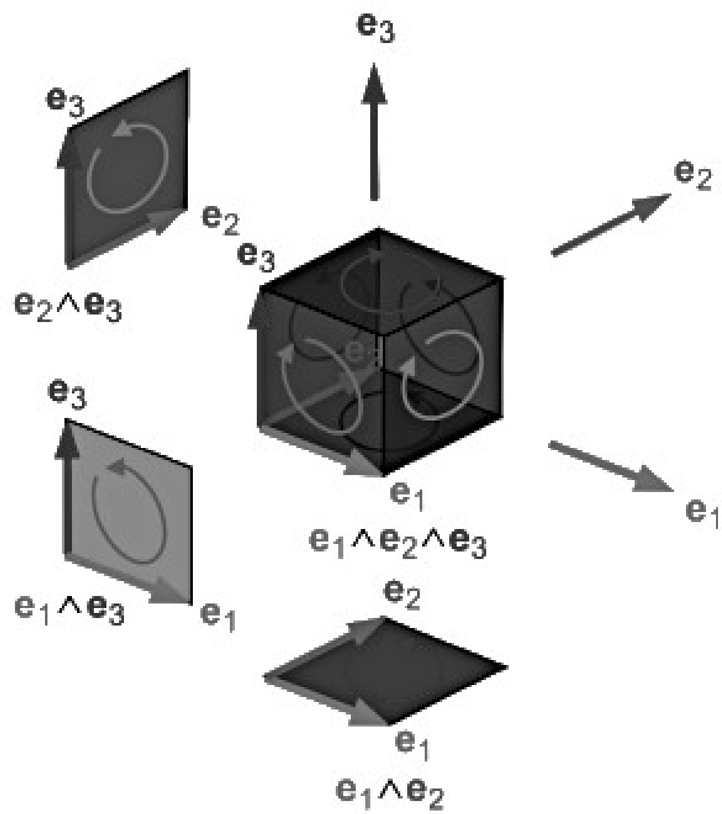


Engineering

- focus on applications and products
- rapidly growing complexity
- need for new tools
  → beyond a black-box (AI/ML)
  → mathematics is a natural choice

This talk

- not difficult to follow the math, but difficult to imagine the applications
- motivate building bridges between engineering and mathematics
  → inspire mathematicians
  → equip engineers with new tools



Part 1: Vector Products

Numbers in more dimensions

Real numbers $\mathcal{R}$

$$x = n + f \equiv f_n, \quad n \in \mathbb{Z},\ f \in [0,1]$$

- n is an integer index of the unit-length boxes
- f is a fractional part (of unit-length box)
- natural total ordering
- form a Group under both addition and multiplication

Group $(G, \circ)$

- binary operation $\circ$ is associative: $a \circ (b \circ c) = (a \circ b) \circ c$
- identity (neutral) element $e \in G$: $a \circ e = e \circ a = a\ \forall a \in G$
- inverse element $b \in G$, for every $a \in G$: $a \circ b = b \circ a = e$

Complex numbers $\mathbb{C} = \mathcal{R}^2$

$$z = x + \mathrm{i}y = (n_x + \mathrm{i}n_y) + (f_x + \mathrm{i}f_y)$$

- $(n_x + \mathrm{i}n_y)$ is a Gaussian integer (or box index)
- $(f_x + \mathrm{i}f_y) \in [0,1]^2$ (unit area 2D box)
- no total ordering
- represent vectors in $\mathcal{R}^2$: $x + \mathrm{i}y = \sqrt{x^2 + y^2}\, \mathrm{e}^{\mathrm{i}\angle(x,y)}$, $\mathrm{i} = \sqrt{-1}$

Numbers in more dimensions (cont.)

Quaternion numbers: $\mathcal{H} = \mathcal{R}^4$

$$h = a + \mathrm{i}b + \mathrm{j}c + \mathrm{k}d = (\underset{\substack{\downarrow \\ \text{scalar part}}}{a}, \underbrace{b, c, d}_{\text{vector part}}) \in \mathcal{R}^4$$

$$-\mathrm{i} = (-1)\mathrm{i} \quad \mathrm{ij} = -\mathrm{ji} = \mathrm{k} \qquad \vec{\mathrm{i}} \triangleq (0,1,0,0)$$

$$-\mathrm{j} = (-1)\mathrm{j} \quad \mathrm{jk} = -\mathrm{kj} = \mathrm{i} \qquad \vec{\mathrm{j}} \triangleq (0,0,1,0)$$

$$-\mathrm{k} = (-1)\mathrm{k} \quad \mathrm{ki} = -\mathrm{ik} = \mathrm{j} \qquad \vec{\mathrm{k}} \triangleq (0,0,0,1)$$

$$\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = \mathrm{ijk} = -1$$

Basic properties

- Hamiltonian product
  → multiply polynomials $\boldsymbol{a} = (a_1 + \mathrm{i}a_2 + \mathrm{j}a_3 + \mathrm{k}a_4)$ and $\boldsymbol{b} = (b_1 + \mathrm{i}b_2 + \mathrm{j}b_3 + \mathrm{k}b_4)$
  → associative, but not commutative
- conjugate

$$\boldsymbol{a} = (a_1, a_2, a_3, a_4) \quad \Rightarrow \quad \boldsymbol{a}^* = (a_1, -a_2, -a_3, -a_4)$$

$$(a_1 + \mathrm{i}a_2 + \mathrm{j}a_3 + \mathrm{k}a_4)^* = a_1 - \mathrm{i}a_2 - \mathrm{j}a_3 - \mathrm{k}a_4$$

$$\boldsymbol{a}^* = -\frac{1}{2}(\boldsymbol{a} + \mathrm{i}\boldsymbol{a}\mathrm{i} + \mathrm{j}\boldsymbol{a}\mathrm{j} + \mathrm{k}\boldsymbol{a}\mathrm{k}) \quad \text{(not valid for complex numbers)}$$

$$(\boldsymbol{ab})^* = \boldsymbol{b}^*\boldsymbol{a}^* \neq \boldsymbol{a}^*\boldsymbol{b}^*$$

- also
  scalar part: $\frac{1}{2}(\boldsymbol{a} + \boldsymbol{a}^*)$, vector part: $\frac{1}{2}(\boldsymbol{a} - \boldsymbol{a}^*)$, $\boldsymbol{ab}^{-1} \neq \boldsymbol{b}^{-1}\boldsymbol{a}$ (division)

Numbers in more dimensions (cont.)

Norms of quaternions

$$||\boldsymbol{a}|| = \sqrt{\boldsymbol{a}\boldsymbol{a}^*} = \sqrt{\boldsymbol{a}^*\boldsymbol{a}} = \sqrt{a_1^2 + a_2^2 + a_3^2 + a_4^2} \Rightarrow \frac{\boldsymbol{a}\boldsymbol{a}^*}{||\boldsymbol{a}||^2} = 1 \Rightarrow \boldsymbol{a}^{-1} = \frac{\boldsymbol{a}^*}{||\boldsymbol{a}||^2}$$

Describing rotations in 3D using quaternions

- pure quaternion: $\mathrm{Re}\{\boldsymbol{u}\} = 0$ (real part)

$$\boldsymbol{u} = (u_x, u_y, u_z) = \mathrm{i}u_x + \mathrm{j}u_y + \mathrm{k}u_z$$

- Euler's rotation theorem: vector $\boldsymbol{u}$ (Euler axis) and (rotation) angle θ

$$\boldsymbol{u}\boldsymbol{u}^* = (0 + \mathrm{i}u_x + \mathrm{j}u_y + \mathrm{k}u_z)(0 - \mathrm{i}u_x - \mathrm{j}u_y - \mathrm{k}u_z) = 1$$

- extension of Euler's formula (Taylor expansion of exp. function)

$$\boldsymbol{q} = \mathrm{e}^{\frac{\theta}{2}\boldsymbol{u}} = \mathrm{e}^{\frac{\theta}{2}(\mathrm{i}u_x + \mathrm{j}u_y + \mathrm{k}u_z)} = \cos\frac{\theta}{2} + \boldsymbol{u}\sin\frac{\theta}{2} \quad \Rightarrow \quad \boldsymbol{q}^{-1} = \mathrm{e}^{-\frac{\theta}{2}\boldsymbol{u}}$$

- to rotate $\boldsymbol{p} = (p_x, p_y, p_z)$ about $\boldsymbol{q}$ by θ to $\boldsymbol{r} = (r_x, r_y, r_z)$, use linear transformation

$$L(\boldsymbol{p}) = \boldsymbol{q}\,(0, \boldsymbol{p})\,\boldsymbol{q}^{-1} = (0, \boldsymbol{r}) \quad \text{(conjugation)}, \quad L(\boldsymbol{q}) = (0, \boldsymbol{q})$$

Dot and cross products of pure quaternions

$$\boldsymbol{a}\cdot\boldsymbol{b} = \frac{1}{2}(\boldsymbol{a}^*\boldsymbol{b} + \boldsymbol{b}^*\boldsymbol{a}) = \frac{1}{2}(\boldsymbol{a}\boldsymbol{b}^* + \boldsymbol{b}\boldsymbol{a}^*), \quad \boldsymbol{a}\times\boldsymbol{b} = \frac{1}{2}(\boldsymbol{a}\boldsymbol{b} - \boldsymbol{b}\boldsymbol{a})$$

Vector (Linear) Spaces

Definition

- a set of vectors that can be scaled by scalars and added together
  → vector elements and scalars $\in \mathcal{F}$ (a field)
- vectors have magnitude and direction
- vector space has finite or countably infinite # dimensions

Axioms of vector spaces

- associativity, commutativity, distributivity
- $\exists$ identity and inverse element

Vector space with additional structures

- algebras
  → linear algebra, polynomial rings, Lie algebras, geometric algebras
- topological vector spaces
  → function spaces, inner product spaces, normed spaces, Hilbert spaces

Key concepts of vector spaces

- linear independence
- linear subspaces (closed under linear combination)
- linear spans (spanning or generating sets of vectors)
- bases (linearly independent vectors spanning sub-spaces)

Vector (Linear) Spaces (cont.)

Hilbert space

- vector space with inner product $\langle \boldsymbol{a}, \boldsymbol{b} \rangle$
  → induces distance $d(\boldsymbol{a}, \boldsymbol{b}) = ||\boldsymbol{a} - \boldsymbol{b}|| = \sqrt{\langle \boldsymbol{a} - \boldsymbol{b}, \boldsymbol{a} - \boldsymbol{b} \rangle}$
- generalizes finite dimen. Euclidean spaces to infinite # dimensions
  → special case of Banach space, e.g. function space: $\langle f, g \rangle = \int f(t) g(t)\, \mathrm{d}t$
- countably infinite dimensions
  → can be described by square-summable infinite sequences

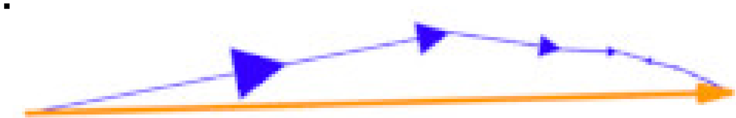
Euclidean space

- special case of Hilbert space
- vectors (Cartesian coordinates) in $\mathcal{R}^n$ with dot-product
  → symmetric, distributative, positive definite

$$\boldsymbol{a} \cdot \boldsymbol{b} = \sum_i a_i b_i = ||\boldsymbol{a}||\,||\boldsymbol{b}|| \cos \theta$$

- absolute convergence of infinite vector sum:

$$\sum_{i=0}^{\infty} \boldsymbol{a}(i) \;\Leftrightarrow\; \sum_{i=0}^{\infty} ||\boldsymbol{a}(i)|| < \infty$$



Applications

- Fourier analysis, eigen-analysis, ODE/PDE, ergodic theory, ...

Vector Products

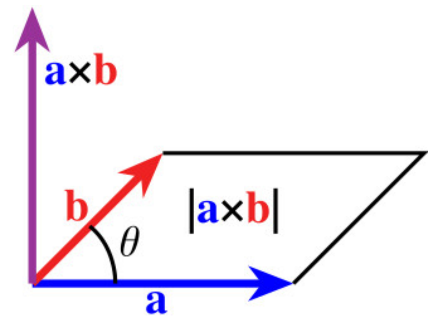
Cross product (in $\mathcal{R}^3$)

- anti-commutative, distributive (over addition), anti-associative

$$\boldsymbol{a}\times\boldsymbol{b} = (a_2b_3 - a_3b_2)\mathrm{i} + (a_3b_1 - a_1b_3)\mathrm{j} + (a_1b_2 - a_2b_1)\mathrm{k}$$

$$\boldsymbol{a}\times\boldsymbol{a} = \boldsymbol{0}, \quad \boldsymbol{a}\times\boldsymbol{b} = -(\boldsymbol{b}\times\boldsymbol{a})$$

$$\boldsymbol{a}\times\boldsymbol{b} = \det\begin{bmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$$



- basis vectors

$$\begin{aligned} &\vec{\mathrm{i}}\times\vec{\mathrm{j}} = \vec{\mathrm{k}}, && \vec{\mathrm{j}}\times\vec{\mathrm{i}} = -\vec{\mathrm{k}}, && \vec{\mathrm{i}}\times\vec{\mathrm{i}} = \boldsymbol{0} \\ &\vec{\mathrm{j}}\times\vec{\mathrm{k}} = \vec{\mathrm{i}}, && \vec{\mathrm{k}}\times\vec{\mathrm{j}} = -\vec{\mathrm{i}}, && \vec{\mathrm{j}}\times\vec{\mathrm{j}} = \boldsymbol{0} \\ &\vec{\mathrm{k}}\times\vec{\mathrm{i}} = \vec{\mathrm{j}}, && \vec{\mathrm{i}}\times\vec{\mathrm{k}} = -\vec{\mathrm{j}}, && \vec{\mathrm{k}}\times\vec{\mathrm{k}} = \boldsymbol{0} \end{aligned}$$

Lie algebra (in $\mathcal{R}^3$)

- e.g. vector space $\mathcal{R}^3$ with vector addition and cross product
- Lie bracket (commutator): $[\boldsymbol{a},\boldsymbol{b}] \triangleq \boldsymbol{a}\times\boldsymbol{b}$
- distributivity: $\boldsymbol{a}\times(\boldsymbol{b}+\boldsymbol{c}) = (\boldsymbol{a}\times\boldsymbol{b}) + (\boldsymbol{a}\times\boldsymbol{c})$
- bi-linearity: $[A\boldsymbol{a}+B\boldsymbol{b},\boldsymbol{c}] = A[\boldsymbol{a},\boldsymbol{c}] + B[\boldsymbol{b},\boldsymbol{c}]$, $A,B\in\mathcal{R}$
- Jacobi identity: $\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c}) + \boldsymbol{b}\cdot(\boldsymbol{c}\times\boldsymbol{a}) + \boldsymbol{c}\cdot(\boldsymbol{a}\times\boldsymbol{b}) = \boldsymbol{0}$

Vector Products (cont.)

Cross product inverse (in $\mathcal{R}^3$)

- given $\boldsymbol{a}, \boldsymbol{c}$, find $\boldsymbol{b}$, so that $\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{c} \quad \Rightarrow \quad \boldsymbol{b} = \frac{1}{\|\boldsymbol{a}\|^2} \boldsymbol{c} \times \boldsymbol{a} + t\,\boldsymbol{a},\ t \in \mathcal{R}$

Linear transformation $\boldsymbol{M} \in \mathcal{R}^3$

$$(\boldsymbol{M}\boldsymbol{a}) \times (\boldsymbol{M}\boldsymbol{b}) = (\det \boldsymbol{M}) \boldsymbol{M}^{-T} (\boldsymbol{a} \times \boldsymbol{b})$$

Rotation invariance about vector (axis) $\boldsymbol{a} \times \boldsymbol{b}$

$$(\boldsymbol{R}\boldsymbol{a}) \times (\boldsymbol{R}\boldsymbol{b}) = \boldsymbol{R}(\boldsymbol{a} \times \boldsymbol{b}),\ \boldsymbol{R}:\ \text{rotation matrix},\ \det \boldsymbol{R} = 1$$

Triple products (in $\mathcal{R}^3$)

$$\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b} \cdot (\boldsymbol{c} \times \boldsymbol{a}) = \boldsymbol{c} \cdot (\boldsymbol{a} \times \boldsymbol{b}) \quad \text{(with absolute value} \triangleq \text{volume)}$$

$$\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{a} \times \boldsymbol{c},\ \boldsymbol{a} \neq \boldsymbol{0} \quad \Rightarrow \quad \underbrace{\boldsymbol{a} \times (\boldsymbol{b} - \boldsymbol{c})}_{\boldsymbol{a} \| (\boldsymbol{b}-\boldsymbol{c})} = \boldsymbol{0} \quad \Rightarrow \quad \boldsymbol{c} = \boldsymbol{b} + t\boldsymbol{a},\ t \in \mathcal{R}$$

$$\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{a} \cdot \boldsymbol{c} \quad \Rightarrow \quad \underbrace{\boldsymbol{a} \cdot (\boldsymbol{b} - \boldsymbol{c})}_{\boldsymbol{a} \perp (\boldsymbol{b}-\boldsymbol{c})} = 0$$

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b}(\boldsymbol{a} \cdot \boldsymbol{c}) - \boldsymbol{c}(\boldsymbol{a}\dot{\boldsymbol{b}}), \quad (\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c} = \boldsymbol{b}(\boldsymbol{c} \cdot \boldsymbol{a}) - \boldsymbol{a}(\boldsymbol{b} \cdot \boldsymbol{c})$$

$$(\boldsymbol{a} \times \boldsymbol{b}) \cdot (\boldsymbol{c} \times \boldsymbol{d}) = (\boldsymbol{a} \cdot \boldsymbol{c})(\boldsymbol{b} \cdot \boldsymbol{d}) - (\boldsymbol{a} \cdot \boldsymbol{d})(\boldsymbol{b} \cdot \boldsymbol{c})$$

Vector Products (cont.)

Norms of vector products (in $\mathcal{R}^3$)

$$\boldsymbol{a}\cdot\boldsymbol{b} = ||\boldsymbol{a}||\,||\boldsymbol{b}||\cos\theta, \quad ||\boldsymbol{a}\times\boldsymbol{b}|| = ||\boldsymbol{a}\wedge\boldsymbol{b}|| = ||\boldsymbol{a}||\,||\boldsymbol{b}||\,|\sin\theta|$$

Lagrange identity

$$||\boldsymbol{a}||^2\,||\boldsymbol{b}||^2 - (\boldsymbol{a}\cdot\boldsymbol{b})^2 = \sum_{1\le i<j\le n}(a_i b_j - a_j b_i)^2,\ n\ge 1 \quad (= ||\boldsymbol{a}\times\boldsymbol{b}||^2,\ n=3$$

Inner product

- associated with inner product (vector) spaces
  $\rightarrow$ inner product induces norm i.e. a normed vector space

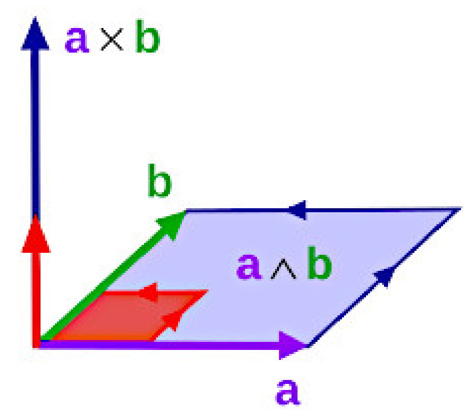
$$\langle\boldsymbol{a},\boldsymbol{b}\rangle = \boldsymbol{b}^{*T}\boldsymbol{a}, \quad \langle f,g\rangle = \int f(t)g^*(t)\,\mathrm{d}t, \quad \langle\boldsymbol{A},\boldsymbol{B}\rangle = \mathrm{tr}\left\{\boldsymbol{A}\boldsymbol{B}^{*T}\right\}$$

- conjugate symmetry (over field $\mathbb{C}$), linearity, positive definite
- can be generalized as Hermitian inner product (over field $\mathbb{C}$)

$$\langle\boldsymbol{a},\boldsymbol{b}\rangle = \boldsymbol{b}^{*T}\boldsymbol{M}\boldsymbol{a}, \quad \boldsymbol{M}:\ \text{Hermitian matrix}$$

Outer (exterior, wedge) product

- generalization of cross-product to $\mathcal{R}^n$, $n>3$
- generalization to multiple vectors
  $\rightarrow$ the product is then a multivector
- e.g.: $\boldsymbol{a}\wedge\boldsymbol{b}$ is a bivector spanned by $\boldsymbol{a}$ and $\boldsymbol{b}$
  $\rightarrow$ oriented surface



Vector Calculus

Scalar and vector fields

- assign scalar or vector to every point in space (-time)
  → space can be a manifold
  → can be generalized to tensor fields (e.g. metric tensor)
- the assignment creates a structure for that space

Pseudovectors vs. true vectors

- induced field may change direction when object or frame of reference are rotated, reflected or otherwise transformed
- examples
  → magnetic field, angular momentum, oriented planes in computer graphics
  → curl of vector field and vector cross product both yield pseudovectors

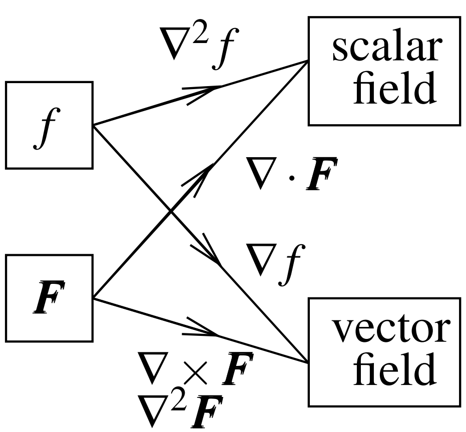
Vector algebra

- vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathcal{R}^3$, and scalar $A \in \mathcal{R}$

$$\boldsymbol{a}+\boldsymbol{b},\ A\boldsymbol{a},\ \boldsymbol{a}\cdot\boldsymbol{b},\ \boldsymbol{a}\times\boldsymbol{b},\ \boldsymbol{c}\cdot(\boldsymbol{a}\times\boldsymbol{b}),\ \boldsymbol{c}\times(\boldsymbol{a}\times\boldsymbol{b})$$

Differential vector operators

- scalar field f, and vector field $\boldsymbol{F}$
  → gradient, divergence, curl, (vector) Laplacian
  → differential forms



Matrix Products

Canonical multiplication

$$\boldsymbol{AB}: \quad \mathcal{R}^{m_1\times n_1}\times\mathcal{R}^{n_1\times n_2}\mapsto\mathcal{R}^{m_1\times n_2}$$

→ systematic collection of dot-products (associative, distributive)

Hadamard product

$$\boldsymbol{A}\odot\boldsymbol{B}: \quad \mathcal{R}^{m\times n}\times\mathcal{R}^{m\times n}\mapsto\mathcal{R}^{m\times n}$$

→ element-wise multiplication (commutative, associative, distributive)

Kronecker product

$$\boldsymbol{A}\otimes\boldsymbol{B}=\begin{bmatrix} a_{11}\boldsymbol{B} & \cdots & a_{1n_1} \\ \vdots & \ddots & \vdots \\ a_{m_1 1}\boldsymbol{B} & \cdots & a_{m_1 n_1}\boldsymbol{B}\end{bmatrix}: \quad \mathcal{R}^{m_1\times n_1}\times\mathcal{R}^{m_2\times n_2}\mapsto\mathcal{R}^{m_1 m_2\times n_1 n_2}$$

→ bilinear, associative, non-commutative

$$(\boldsymbol{A}\otimes\boldsymbol{B})^{-1}=\boldsymbol{A}^{-1}\otimes\boldsymbol{B}^{-1}, \quad (\boldsymbol{A}\otimes\boldsymbol{B})^T=\boldsymbol{A}^T\otimes\boldsymbol{B}^T, \quad \det(\boldsymbol{A}\otimes\boldsymbol{B})=(\det\boldsymbol{A})^m(\det\boldsymbol{B})^n$$

$$\boldsymbol{A}\oplus\boldsymbol{B}=\boldsymbol{A}\otimes\boldsymbol{I}_m+\boldsymbol{I}_n\otimes\boldsymbol{B} \quad \text{(Kronecker sum)}$$

Mixed products

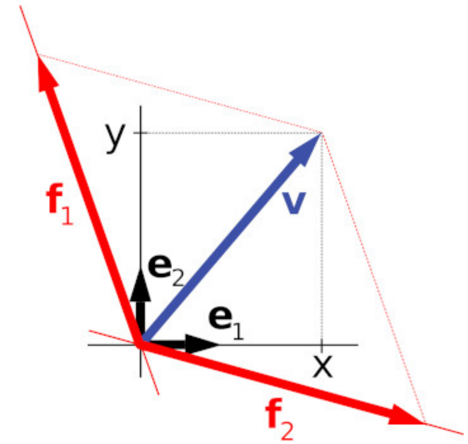
$$(\boldsymbol{A}\otimes\boldsymbol{B})(\boldsymbol{C}\otimes\boldsymbol{D})=(\boldsymbol{AC})\otimes(\boldsymbol{BD})$$

$$(\boldsymbol{A}\otimes\boldsymbol{B})\odot(\boldsymbol{C}\otimes\boldsymbol{D})=(\boldsymbol{A}\odot\boldsymbol{C})\otimes(\boldsymbol{B}\odot\boldsymbol{D})$$

Frobenius inner product

$$\langle\boldsymbol{AB}\rangle_F=\mathrm{tr}\left\{\boldsymbol{A}^T\boldsymbol{B}\right\}$$

Tensors



Multi-dimensional arrays?

- yes, but one (very narrow) interpretation

Geometric vectors?

- magnitude & direction the same in different bases
- rank 1 tensor, contravariant vector

Key properties

- tensor can be represented as ordered list of numbers (vector) in given basis
- object represented by a tensor does not change in different bases
  $\rightarrow$ not every matrix is a tensor
- tensor rank (order, degree) is # dimensions of the object it represents

Contravariant vector $(1,0)$-tensor

- basis are columns of $\boldsymbol{B}$, so $\boldsymbol{v} = \boldsymbol{B} \cdot \tilde{\boldsymbol{v}}$
- basis rotation & scaling via $\boldsymbol{T}$

$$\boldsymbol{v} = \underbrace{\boldsymbol{BT}}_{\text{basis}} \cdot \underbrace{\boldsymbol{T}^{-1}\tilde{\boldsymbol{v}}}_{\text{components}}$$

Covariant vector (covector) $(0,1)$-tensor

- co-varies with basis transformation
- it is a linear function $f(\boldsymbol{x}) = \langle \boldsymbol{v}, \boldsymbol{x} \rangle$
- value $f(\boldsymbol{x})$ is independent of basis

Tensors (Cont.)

Linear transformation $(1,1)$-tensor

- change of basis: $\tilde{\boldsymbol{y}} = \boldsymbol{T}\boldsymbol{y}$ and $\tilde{\boldsymbol{x}} = \boldsymbol{T}\boldsymbol{x}$
- i.e., if $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$, then $\tilde{\boldsymbol{y}} = \tilde{\boldsymbol{A}}\tilde{\boldsymbol{x}}$ where $\tilde{\boldsymbol{A}} = \boldsymbol{T}\boldsymbol{A}\boldsymbol{T}^{-1}$
  $\rightarrow \boldsymbol{T}^{-1}$ is contravariant
  $\rightarrow \boldsymbol{T}$ is covariant
  $\rightarrow \boldsymbol{T}\boldsymbol{A}\boldsymbol{T}^{-1}$ is $(1,1)$-tensor, i.e., rank 2 tensor (2×2 matrix)

Bilinear transformation $B: \ \boldsymbol{u},\boldsymbol{v} \mapsto \mathcal{R}$

$$\begin{aligned} B(\boldsymbol{u}+\boldsymbol{w},\boldsymbol{v}) &= B(\boldsymbol{u},\boldsymbol{v}) + B(\boldsymbol{w},\boldsymbol{v}) \\ B(\lambda\boldsymbol{u},\boldsymbol{v}) &= \lambda B(\boldsymbol{u},\boldsymbol{v}) \\ B(\boldsymbol{u},\boldsymbol{v}+\boldsymbol{w}) &= B(\boldsymbol{u},\boldsymbol{v}) + B(\boldsymbol{u},\boldsymbol{w}) \\ B(\boldsymbol{u},\lambda\boldsymbol{v}) &= \lambda B(\boldsymbol{u},\boldsymbol{v}) \end{aligned} \quad \Rightarrow \quad B(\boldsymbol{u},\boldsymbol{v}) = \boldsymbol{u}^T\boldsymbol{A}\boldsymbol{v} = \sum_{i,j=1}^{n} a_{i,j}u_i v_j = A_{ij}u^i v^j$$

- $\boldsymbol{A}$ is rank $(0,2)$-tensor (with two covectors)
- $\boldsymbol{u}$ and $\boldsymbol{v}$ are $(1,0)$-tensors (contravariants)
- with transformation of basis $\boldsymbol{T}$, $\tilde{A}_{ij} = A_{ij}T^i_k T^j_l$

Tensors (Cont.)

Rank n tensor in $\mathcal{R}^m$

- have n indices, $1 \leq i \leq m$, and m^n components
  $\rightarrow$ plus certain structure defined by transformation rules
- generalization of matrices, e.g. in $\mathcal{R}^3$

$$\boldsymbol{A} = [a_{ijk}] \text{ (matrix)} \quad \longrightarrow \quad \boldsymbol{A} = \left[a_{ijk} \text{ or } a_{ij}^{\ \ k} \text{ or } a_i^{\ jk} \text{ or } a^{ijk} \ \cdots \right] \text{ (tensor)}$$

Einstein's summation convention

- repeated indices are summed over
- each index can appear at most twice
- each term must contain identical non-repeated indices
- index lowering and index raising:

$$g^{ij}A_j = A^i, \quad g_{ij}A^j = A_i \quad (g: \text{ metric tensor})$$

- dot and cross products

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_i b^j, \quad (\boldsymbol{a} \times \boldsymbol{b})_i = \epsilon_{ijk} a^j b^k$$

$$\epsilon_{ijk} = \vec{\mathrm{i}} \cdot (\vec{\mathrm{j}} \times \vec{\mathrm{k}}) = [\mathrm{i}, \mathrm{j}, \mathrm{k}] \text{ (permutation tensor)}$$

$$a_i a_i \triangleq \sum_i a_i a_i$$

$$a_{ik} a_{ij} \triangleq \sum_i a_{ik} a_{ij}$$

$$A_{ij} b_j \triangleq \sum_j A_{ij} b_j$$

Tensors (Cont.)

Summing tensors

- must have the same rank and the same indices, e.g., rank-2 tensors

$$A^{ij}+B^{ij}, \quad A_{ij}+B_{ij}, \quad ,A_i^{\,j}+B_i^{\,j}, \quad A^i{}_j+B^i{}_j$$

Dot-product of tensors

- known as tensor contraction
  → set unlike indices equal and then sum using Einstein summation
- tensor rank reduced by 2, e.g. rank-2 tensor

$$\mathrm{contr}\left(T^i{}_j\right)=T^i{}_i\equiv\sum_i T^i{}_i\in\mathcal{R}$$

Tensor product

- product between two vector spaces A and B over the same field
- it is a bilinear map:

$$A\times B\mapsto A\otimes B \;\Rightarrow\; (\boldsymbol{a}\in A,\boldsymbol{b}\in B)\mapsto(\boldsymbol{a}\otimes\boldsymbol{b})\in A\otimes B$$

- $(\boldsymbol{a}\otimes\boldsymbol{b})$ is a decomposable tensor
  → e.g. product of rank-1 tensors: $\boldsymbol{a}\otimes\boldsymbol{b}=\boldsymbol{a}\boldsymbol{b}^T$ (matrix)
- applications
  → $A\otimes B\mapsto C$ can be uniquely factored into two linear maps
  → metric tensor is created as product of the base tensor with itself

Take-Home Messages

Mathematical objects

- functions, vectors, matrices
- numbers in $\mathcal{R}^n$, for $n = 1, 2, 4$:
  → real, complex, quaternions)
- multivectors, tensors
- vector spaces
  Euclidean, Hilbert, Banach, ...
- sets, graphs, manifolds
- groups, fields, rings

Manipulating math objects

$$\langle \text{Object}_1 \rangle \ (operation) \ \langle \text{Object}_2 \rangle \ \longrightarrow \ \langle \text{Object}_3 \rangle$$

$$(Operator) \ \langle \text{Object}_1 \rangle \ \longrightarrow \ \langle \text{Object}_2 \rangle$$

- often group, i.e. $\text{Object}_i \in \text{Group}$, for $\forall i$
- algebras (linear, vector, Lie, Clifford, ...)
  → algebraic operations (especially addition and multiplication)
- calculus
  → integral, differential, vector calculus
- computing
  → assign numerical values to math objects

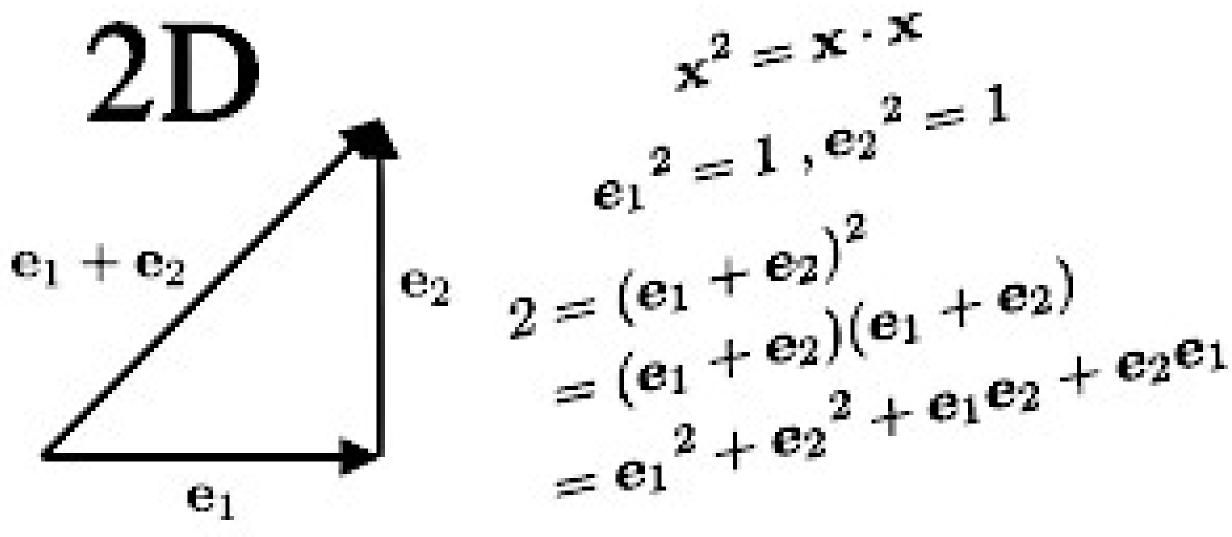
Take-Home Messages (cont.)

Vector products

- inner product
  → dot product
- outer product
  → exterior, wedge, cross products
- combined (triple) products
- matrix products
  → canonical, Hadamard, Kronecker, ...
- tensor product
- geometric product
  → geometric algebra

Product properties

- associative, commutative, distributive
  → anti-commutative, anti-associative



Part 2: Geometric Algebra

GEOMETRIC ALGEBRA

Geometric algebra

- geometric properties
- focus on applications

Clifford algebra

- mathematical properties
- focus on abstractions

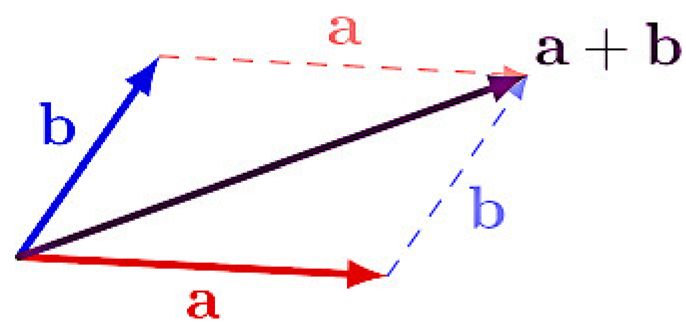
Scalers

- scalars, 0-dimensional, manipulated via algebra for real numbers

Vectors

- 1-dimensional, vector algebra including scaling and adding vectors
- all vectors with the same magnitude (length) and direction are equal
  → directions may differ in higher-dimensional spaces
- decomposition into a basis of (orthogonal) unit vectors

$$\vec{a} = \sum_i a_i \vec{e}_i, \quad \vec{a} = a_1 \vec{\imath} + a_2 \vec{\jmath} + a_3 \vec{k} \text{ (in } \mathcal{R}^3\text{)}$$

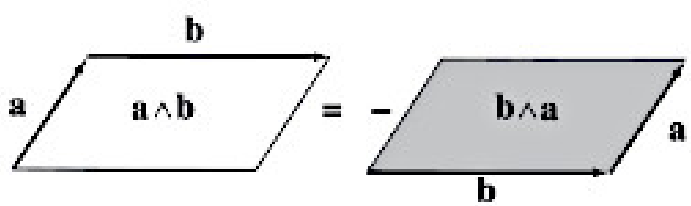


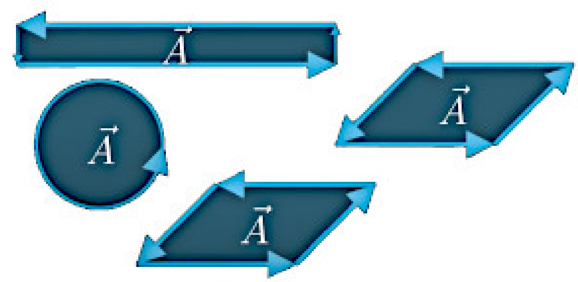
Geometric Algebra (cont.)

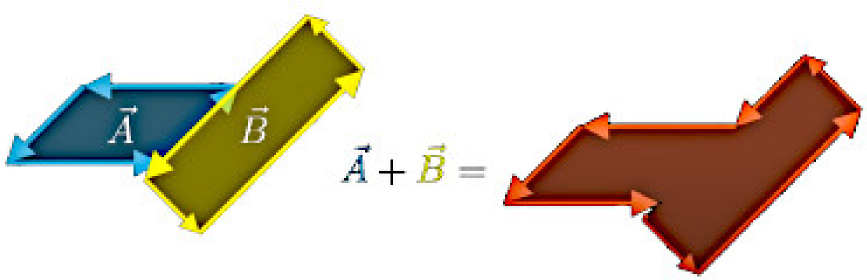
Bivectors

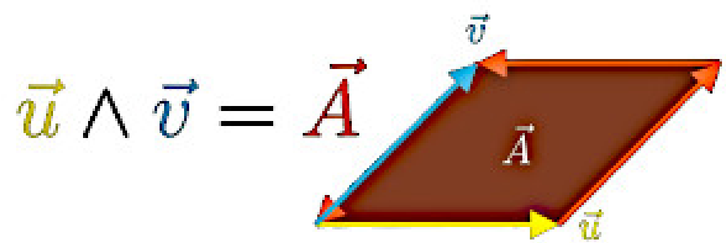
- 2-dimensional oriented surface (bivector magnitude == surface area)
- all bivectors with the same magnitude and orientation are equal
  $\rightarrow$ more tricky in higher-dimensional spaces
- bivectors can be morphed without changing magnitude and orientation
- morphing enables bivector addition in higher dimensions
  $\rightarrow$ scale and morph them before adding them together
- decomposition into a basis of (orthogonal) unit bivectors

$$\vec{\boldsymbol{A}} = A_1\vec{I} + A_2\vec{J} + A_3\vec{K} \quad (\text{in } \mathcal{R}^3)$$





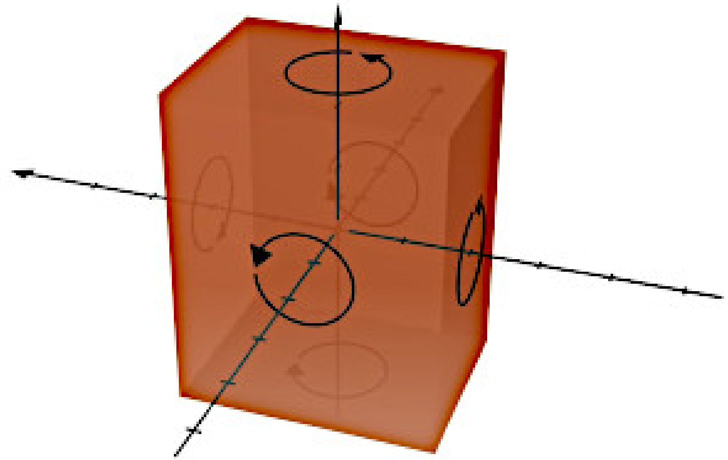




Geometric Algebra (cont.)

Trivectors

- 3-dimensional oriented volumes
- magnitude == volume size
- decomposition into unit trivectors
- can be generalized to k-vectors

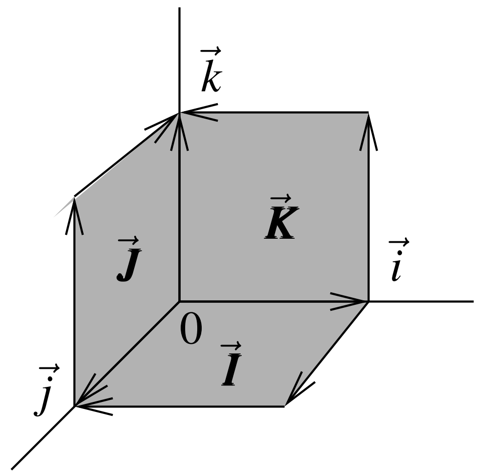


any order k-vectors can be summed

Outer products

- vector ∧ bivector = trivector
- product of basis vectors:

$$\vec{I} = \vec{\mathrm{i}} \wedge \vec{\mathrm{j}}, \quad \vec{J} = \vec{\mathrm{j}} \wedge \vec{\mathrm{k}}, \quad \vec{K} = \vec{\mathrm{i}} \wedge \vec{\mathrm{k}}$$



Geometric product

- key concept of geometric algebra

$$\vec{a}\vec{b} = \underbrace{\vec{a} \cdot \vec{b}}_{\text{scaler}} + \underbrace{\vec{a} \wedge \vec{b}}_{\text{bivector}} = (a_1\mathrm{i} + a_2\mathrm{j} + a_3\mathrm{k})(b_1\mathrm{i} + b_2\mathrm{j} + b_3\mathrm{k}) \quad \text{c.f.} \quad A + \mathrm{i}B \in \mathbb{C}$$

Geometric Product

$$\vec{a}\vec{b} = \vec{a}\cdot\vec{b} + \vec{a}\wedge\vec{b}$$

Properties

- vectors can be divided

$$\vec{a}\vec{a} = \underbrace{\vec{a}\cdot\vec{a}}_{\|\vec{a}\|^2} + \underbrace{\vec{a}\wedge\vec{a}}_{0} = \|\vec{a}\|^2 \quad\Rightarrow\quad \boxed{\vec{a}^2 = \|\vec{a}\|^2} \quad\Rightarrow\quad \vec{a}^{-1} = \frac{\vec{a}}{\|\vec{a}\|^2}$$

- swapping vectors

$$\vec{b}\vec{a} = \vec{a}\cdot\vec{b} - \vec{b}\wedge\vec{a} \quad\Rightarrow\quad \boxed{\begin{array}{rcl} \vec{a}\cdot\vec{b} &=& \frac{1}{2}\left(\vec{a}\vec{b}+\vec{b}\vec{a}\right) \\ \vec{a}\wedge\vec{b} &=& \frac{1}{2}\left(\vec{a}\vec{b}-\vec{b}\vec{a}\right) \end{array}} \begin{array}{l}\text{(inner product)}\\ \text{(outer product)}\end{array}$$

- basis vectors

$$\mathrm{i}^2 = \vec{\imath}\vec{\imath} = \|\vec{\imath}\|^2 = 1, \quad \mathrm{j}^2 = \vec{\jmath}\vec{\jmath} = \|\vec{\jmath}\|^2 = 1, \quad \mathrm{k}^2 = \vec{\mathrm{k}}\vec{\mathrm{k}} = \|\vec{\mathrm{k}}\|^2 = 1$$

$$\vec{\imath}\perp\vec{\jmath}\perp\vec{\mathrm{k}} \;\Rightarrow\; \vec{\imath}\cdot\vec{\jmath} = \vec{\imath}\cdot\vec{\mathrm{k}} = \vec{\jmath}\cdot\vec{\mathrm{k}} = 0 \quad\Rightarrow\quad \boxed{\begin{array}{rcccl} \vec{\imath}\vec{\jmath} &=& \vec{\imath}\wedge\vec{\jmath} &=& -\vec{\jmath}\vec{\imath} \\ \vec{\imath}\vec{\mathrm{k}} &=& \vec{\imath}\wedge\vec{\mathrm{k}} &=& -\vec{\mathrm{k}}\vec{\imath} \\ \vec{\jmath}\vec{\mathrm{k}} &=& \vec{\jmath}\wedge\vec{\mathrm{k}} &=& -\vec{\mathrm{k}}\vec{\jmath} \end{array}}$$

Geometric Product (cont.)

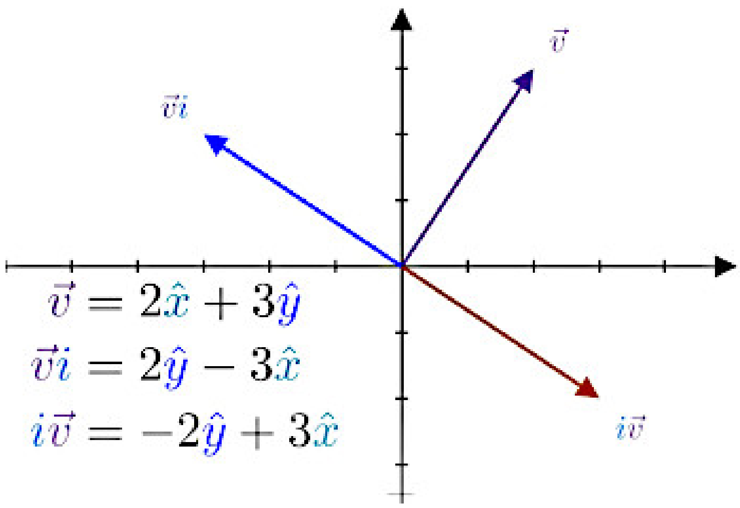
General procedure (for multiplying any k-vectors)

1. express vectors in terms of basis vectors $\vec{e}_i$
2. multiply the vectors as polynomials by distributing all terms
3. simplify the expressions using $\vec{e}_i\vec{e}_i = 1$ and $\vec{e}_i\vec{e}_j = -\vec{e}_j\vec{e}_i$

$(k > 0)$-vectors

- linear combinations of other k-vectors
- e.g. vectors in $\mathcal{R}^2$:

$$\boldsymbol{v} = \underbrace{v_0}_{\text{scaler}} + \underbrace{v_1\vec{e}_1 + v_2\vec{e}_2}_{\text{vectors}} + \underbrace{v_3\vec{e}_1\vec{e}_2}_{\text{bivector}}$$



- $\vec{e}_1\vec{e}_2$ is a pseudo-scaler
  $\rightarrow \vec{e}_1\vec{e}_2 \triangleq \mathrm{i} \;\Rightarrow\; \mathrm{i}^2 = -1$
  $\rightarrow$ thus, $\vec{\boldsymbol{v}}\mathrm{i}$ and $\mathrm{i}\vec{\boldsymbol{v}}$ are 90° rotations
  $\rightarrow$ $\vec{\boldsymbol{v}}z = z^*\vec{\boldsymbol{v}}$ are arbitrary rotations

Complex numbers

$$\underbrace{a}_{\text{scaler}} + \underbrace{\mathrm{i}b}_{\text{pseudo-scaler}} \in \mathbb{C}$$

$$\begin{aligned}
\vec{\boldsymbol{a}}\vec{\boldsymbol{b}} &= \vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{b}} + \vec{\boldsymbol{a}}\wedge\vec{\boldsymbol{b}} \\
(a+c\mathrm{i})(c+d\mathrm{i}) &= (ac-bd)+(ad+bc)\mathrm{i} \\
(a\vec{e}_1+b\vec{e}_2)(c\vec{e}_1+d\vec{e}_2) &= (ac-bd)\vec{e}_1+(ad+bc)\vec{e}_2
\end{aligned}$$

Geometric Product (cont.)

Rotations

- unit vectors $\boldsymbol{a}$ and $\boldsymbol{b}$
  $\boldsymbol{ab} = \boldsymbol{a}\cdot\boldsymbol{b} + \boldsymbol{a}\wedge\boldsymbol{b} = ||\boldsymbol{a}||\,||\boldsymbol{b}||\cos\theta + ||\boldsymbol{a}||\,||\boldsymbol{b}||\sin\theta\,\mathrm{i} = \mathrm{e}^{\mathrm{i}\theta}$, and, $\boldsymbol{ba} = (\boldsymbol{ab})^* = \mathrm{e}^{-\mathrm{i}\theta}$
- thus, the rotation of any vector $\vec{\boldsymbol{c}}$

$$\left.\begin{array}{l}\vec{\boldsymbol{c}}\boldsymbol{ab} = \vec{\boldsymbol{c}}\,\mathrm{e}^{ii\theta} \\ \vec{\boldsymbol{c}}\boldsymbol{ba} = \vec{\boldsymbol{c}}\,\mathrm{e}^{-\mathrm{i}\theta}\end{array}\right\} \quad\Rightarrow\quad \vec{\boldsymbol{c}}\vec{\boldsymbol{a}}\vec{\boldsymbol{b}} = \vec{\boldsymbol{b}}\vec{\boldsymbol{a}}\vec{\boldsymbol{c}}$$

Extension to $\mathcal{R}^3$

$$\boldsymbol{v} = \underbrace{a_1}_{\text{scaler}} + \underbrace{a_2\vec{e}_1 + a_3\vec{e}_2}_{\text{vectors}} + \underbrace{a_4\vec{e}_1\vec{e}_2}_{\text{bivector}} \in \mathcal{R}^2$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$\boldsymbol{v} = \underbrace{a_1}_{\text{scaler}} + \underbrace{a_2\vec{e}_1 + a_3\vec{e}_2 + a_4\vec{e}_3}_{\text{vectors}} + \underbrace{a_5\vec{e}_1\vec{e}_2 + a_6\vec{e}_1\vec{e}_3 + a_7\vec{e}_2\vec{e}_3}_{\text{bivectors}} + \underbrace{a_8\vec{e}_1\vec{e}_2\vec{e}_3}_{\text{trivector}} \in \mathcal{R}^3$$

- $a_8\vec{e}_1\vec{e}_2\vec{e}_3 \triangleq a_8\mathrm{i}$ is a pseudo-scaler, and again, $\mathrm{i}^2 = -1$

Geometric Product (cont.)

Properties in $\mathcal{R}^3$

- multiplying by $\boxed{\mathrm{i} = \vec{e}_1\vec{e}_2\vec{e}_3}$
  → i commutes with any 3-vector:
  $$\mathrm{i}A = A\mathrm{i}, \quad \text{vector} \underset{\mathrm{i}}{\overset{\mathrm{i}}{\rightleftarrows}} \text{bivector}$$

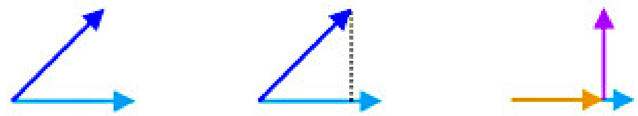
$$\begin{aligned} \mathrm{i}\vec{e}_1 &= \vec{e}_2\vec{e}_3 & & \vec{e}_1 \perp \vec{e}_2\vec{e}_3 \\ \mathrm{i}\vec{e}_2 &= \vec{e}_1\vec{e}_3 & \Rightarrow \quad & \vec{e}_2 \perp \vec{e}_1\vec{e}_3 \\ \mathrm{i}\vec{e}_3 &= \vec{e}_1\vec{e}_2 & & \vec{e}_3 \perp \vec{e}_1\vec{e}_2 \end{aligned}$$

$$\boxed{\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = \mathrm{ijk} = -1}$$

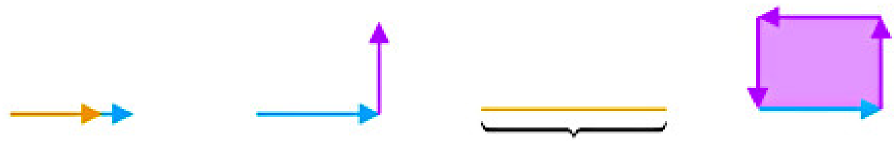
- bivector can be represented by its normal vector
  → 3-vectors are scalers and vectors: $a + b\mathrm{i} + \vec{\boldsymbol{a}} + \vec{\boldsymbol{b}}\mathrm{i}$
- bivectors = pseudovectors: $$\underbrace{\vec{\boldsymbol{a}} \wedge \vec{\boldsymbol{b}}}_{\text{bivector}} = \mathrm{i}\,\underbrace{\vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}}}_{\text{vector}}$$
  → e.g. for vector field $\boldsymbol{F}$: $\vec{\nabla} \wedge \vec{\boldsymbol{F}} = \mathrm{i}\vec{\nabla} \times \vec{\boldsymbol{F}}$
  → pseudovector basis: $\{\vec{e}_1\vec{e}_2, \vec{e}_2\vec{e}_3, \vec{e}_1\vec{e}_3\}$
- trivectors = pseudoscalars: $\vec{\boldsymbol{a}} \cdot (\vec{\boldsymbol{b}} \times \vec{\boldsymbol{c}}) = \mathrm{i}\,\vec{\boldsymbol{a}} \wedge \vec{\boldsymbol{b}} \wedge \vec{\boldsymbol{c}}$
- moreover
  $$\begin{aligned} \text{in } \mathcal{R}^3 : \text{scaler} + \text{bivector} \quad &\triangleq \quad \text{quaternion} \\ \text{in } \mathcal{R}^2 : \text{scaler} + \text{bivector} \quad &\triangleq \quad \text{complex number} \end{aligned}$$
- to rotate $\vec{\boldsymbol{a}}$ by θ in plane $\vec{\boldsymbol{B}}$: $$\mathrm{e}^{-\vec{\boldsymbol{B}}\frac{\theta}{2}}\,\vec{\boldsymbol{a}}\,\mathrm{e}^{\vec{\boldsymbol{B}}\frac{\theta}{2}} \triangleq \text{rotor}^*\,\vec{\boldsymbol{a}}\,\text{rotor}$$

Geometric Product – Summary

Multiplying two vectors



$$\vec{u}\vec{v} = \vec{u}\vec{v} = \vec{u}(\vec{v}_{\|} + \vec{v}_{\perp})$$



$$= \vec{u}\vec{v}_{\|} + \vec{u}\vec{v}_{\perp} = \vec{u}\cdot\vec{v} + \vec{u}\wedge\vec{v}$$

- $\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}}$ is inner product (scaler)
- $\vec{\boldsymbol{u}} \wedge \vec{\boldsymbol{v}}$ is outer product (bivector)

General rules for multiplying k-vectors in any dimensions

- extract parallel and perpendicular components
  $\rightarrow$ inner product: $\|$ are multiplied, $\perp$ cancels out
  $\rightarrow$ outer product: $\|$ cancels out, $\perp$ join into higher-dimensional k-vector
- examples:

$$\begin{aligned}
\text{(vector)}\,\text{(vector)} &= \text{scaler} + \text{bivector} \\
\text{(vector)}\,\text{(bivector)} &= \text{vector} + \text{trivector} \\
\text{(bivector)}\,\text{(bivector)} &= \underbrace{\text{scalar}}_{\text{inner product}} + \text{bivector} + \underbrace{4-\text{vector}}_{\text{outer product}}
\end{aligned}$$

Take-Home Messages

Geometric algebra

- key concepts
  → multivectors and their (geometric) product in $\mathcal{R}^n$ vector spaces
- multiplication allows defining (linear) maps and functions
- not all properties translate across dimensions ($\mathcal{R}^2$ and $\mathcal{R}^3$ most relevant)
- basic applications
  → rotations, reflections, translations in $\mathcal{R}^n$

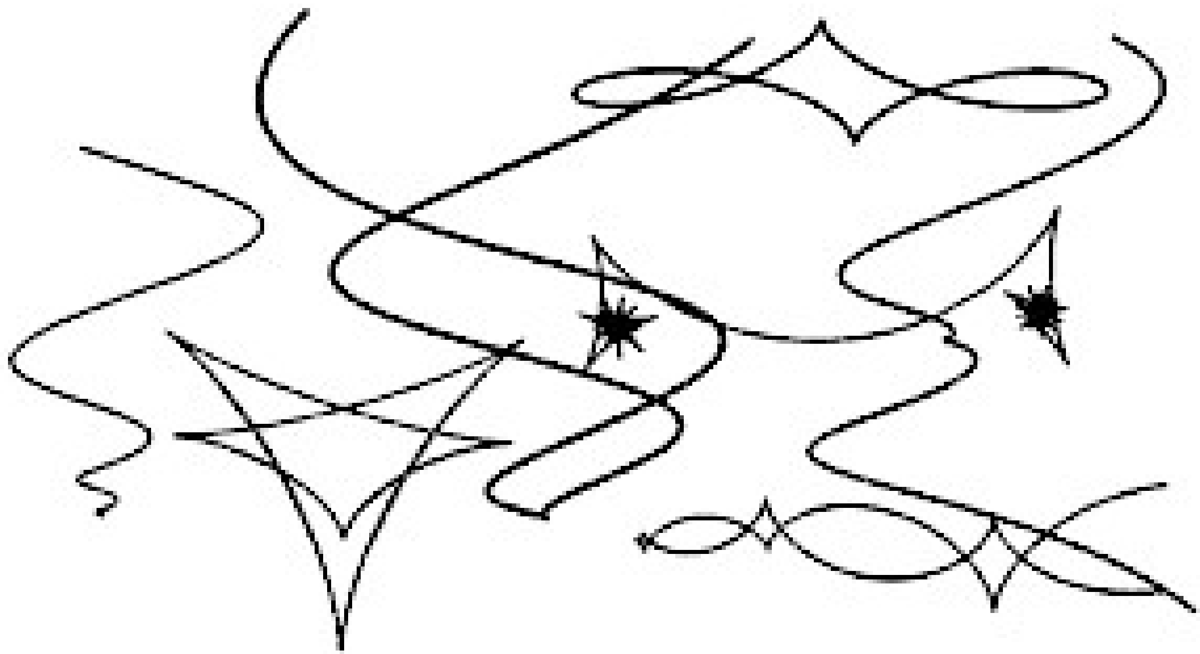
In general

- GA is a powerful and efficient modeling language
- GA can bring new insights and connections
- there are several different versions of GA

To remember (in $\mathcal{R}^3$)

- $\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = \mathrm{ijk} = -1$
- $\vec{a}\vec{b} = \underbrace{\vec{a} \cdot \vec{b}}_{\text{inner product}} + \underbrace{\vec{a} \wedge \vec{b}}_{\text{outer product}}$
- $\vec{a}^2 = \|\vec{a}\|^2$
- $\vec{\imath}\vec{a}$ and $\vec{a}\vec{\imath}$ are 90° rotations

$$
\begin{aligned}
\text{(scalar) (vector)} &= \text{(vector)} \\
\text{(vector) (vector)} &= \text{scaler} + \text{bivector} \\
\text{(vector) (bivector)} &= \text{vector} + \text{trivector} \\
\text{(bivector) (bivector)} &= \text{scalar} + 2\text{−vect} + 4\text{−vect}
\end{aligned}
$$



Part 3: Curves

Bézier Curves

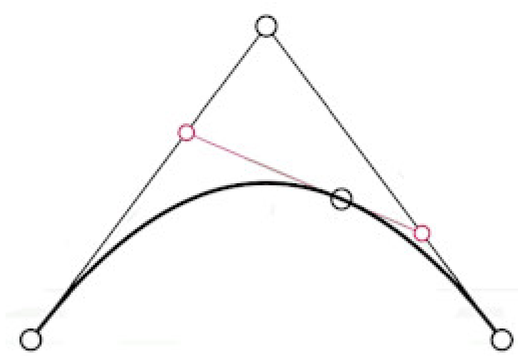
Curves

- describe a *smooth* trajectory between point P_0 and point P_1
  → typographic fonts, computer games, simulations, non-linear functions
- here, let's focus on parameterized curves in 2D
  → parameters can be optimized or learned

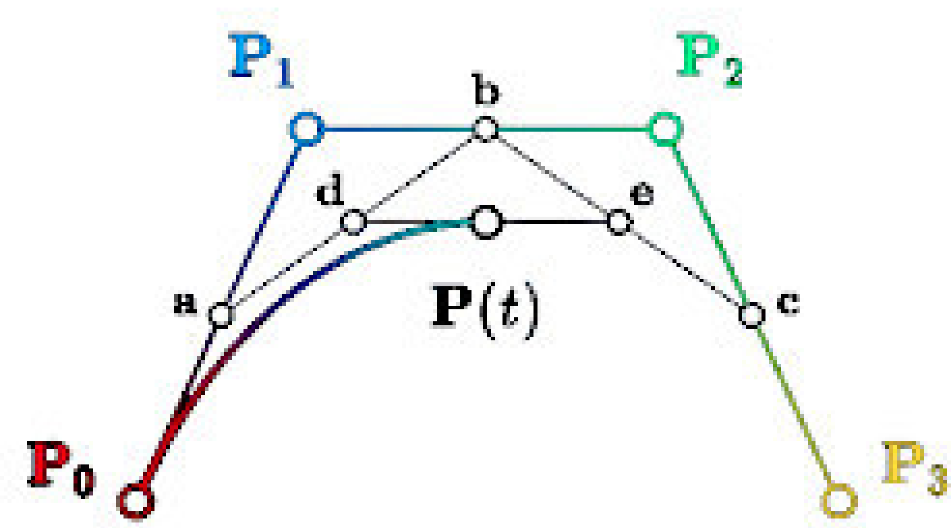
Definition of Bézier curves

- linear interpolation (Lerp)

$$P(t) = (1-t)P_0 + tP_1 \triangleq \mathrm{lerp}(P_0, P_1, t), \quad 0 \le t \le 1$$



- cubic Bézier curves are most common
  → used for re-scaling images, fonts etc.



$$a = \mathrm{lerp}(P_0, P_1, t) \qquad d = \mathrm{lerp}(a, b, t)$$
$$b = \mathrm{lerp}(P_1, P_2, t) \qquad e = \mathrm{lerp}(b, c, t)$$
$$c = \mathrm{lerp}(P2, P_3, t) \qquad P = \mathrm{lerp}(d, e, t)$$

$$\begin{aligned} P = \quad & P_0(-t^3 + 3t^2 - 3t + 1) + \\ & P_1(3t^3 - 6t^2 + 3t) + \\ & P_2(-3t^3 + 3t^2) + \\ & P_3(t^3) \end{aligned}$$

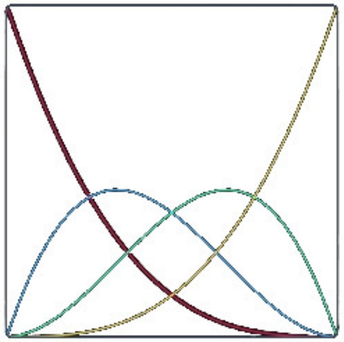
Bézier Curves (cont.)

Cubic Bézier curve

$$P = P_0(-t^3+3t^2-3t+1) + P_1(3t^3-6t^2+3t) + P_2(-3t^3+3t^2) + P_3(t^3)$$

$$P(t) = \sum_{i=0}^{3} P_i\, p_i(t)$$

$$\sum_{i=0}^{3} p_i(t) \overset{!}{=} 1 \ \forall t$$

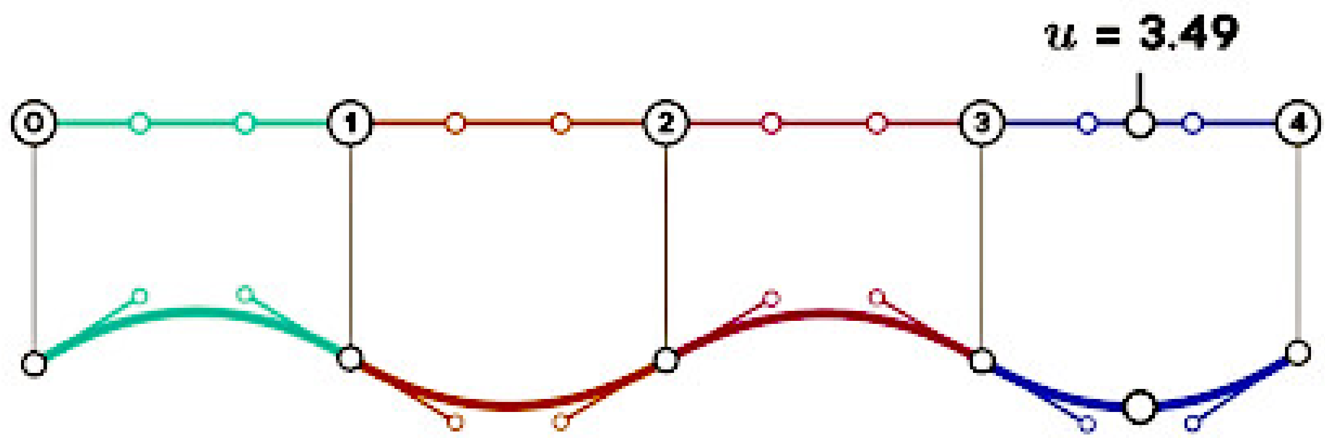


$$P(t) = P_0 + t(-3P_0+3P_1) + t^2(3P_0-6P_1+3P_2) + t^3(-P_0+3P_1-3P_2+P_3)$$

$$P(t) = \begin{bmatrix} 1 & t & t^2 & t^3 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}}_{\text{characteristic matrix}} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}$$

- note that (control) points P_i are (2D) vectors
- different representations of the same curve
  → may differ in numerical efficiency and numerical stability
- can be generalized to any higher degrees (in 2D)
  → becomes very ineffective in controlling the curve shape
  → no local control, numerically complex and unstable

Bézier Splines

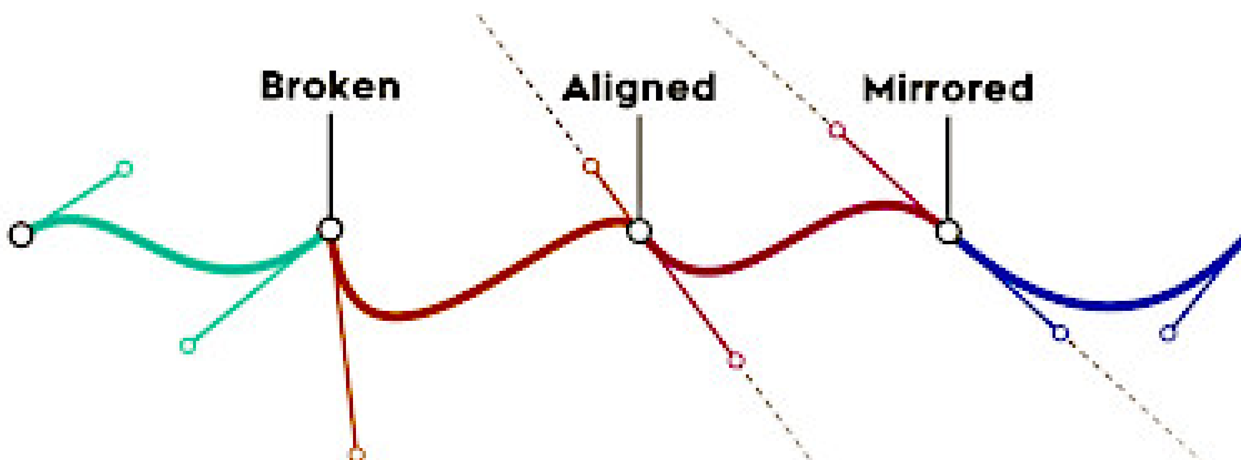


Definition

- piecewise cubic Bézier splines
  → defined by individual control points
- pieces connect at joins (knots)
  → knot intervals (length of pieces)

Advantages

- full local control
- easy to add more segments
- num. efficiency and stability
- interpolate every 3rd point



→ mirrored control points are sufficient for continuous velocity

Curves Continuity

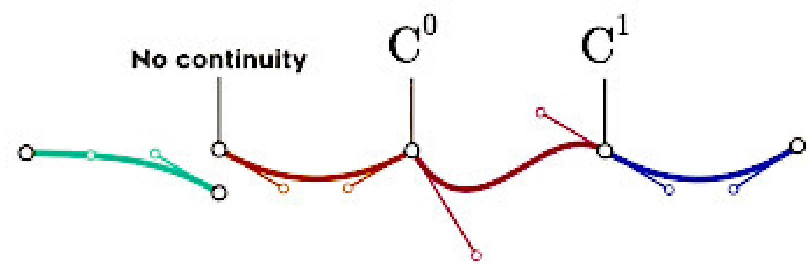
Parametric continuity

C^0:	$P(t)$	position
C^1:	$P'(t)$	velocity
C^2:	$P''(t)$	jolt

C^i continuity implies continuities $C^{i-1}, \ldots, C^0$

Caveats

- the more continuities, the less control
  → also control sensitivity greatly increased
- for cubic splines
  → all 3-rd and higher derivatives are zero
  → C^2 continuity looses most control



Geometric continuity

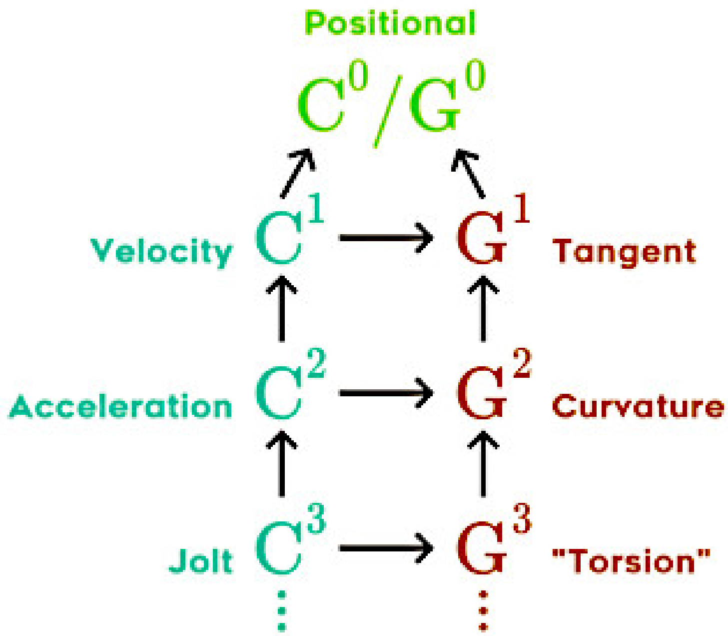
- parameter-free and more control freedom than C-continuities
- tangent continuity: $P'(u)/\,\|P'(u)\|$
  → equivalent to G^1 continuity (aligning left and right tangent vectors)
- G^2 continuity is evaluated as a curvature
  → it is $1/\text{radius}$ of the circle locally approximating the curve
- $A(t)$ and $B(t)$ are G^n continuous, if

$$A(t) \text{ and } B(g(t)) \text{ are } C^n \text{ continuous for some function } g(t)$$

Curves Continuity (cont.)

Regular curves

- the curves with $P'(t) \neq 0$ for $\forall t \geq 0$



Other curves

- Hermite splines
  → specify start and end positions *and* velocities
  → defined as C^1 continuous
- piecewise linear function is C^0
  → can be modified into cardinal spline to get C^1
  → special case of cardinal spline is Catmull-Rom spline

B-Splines

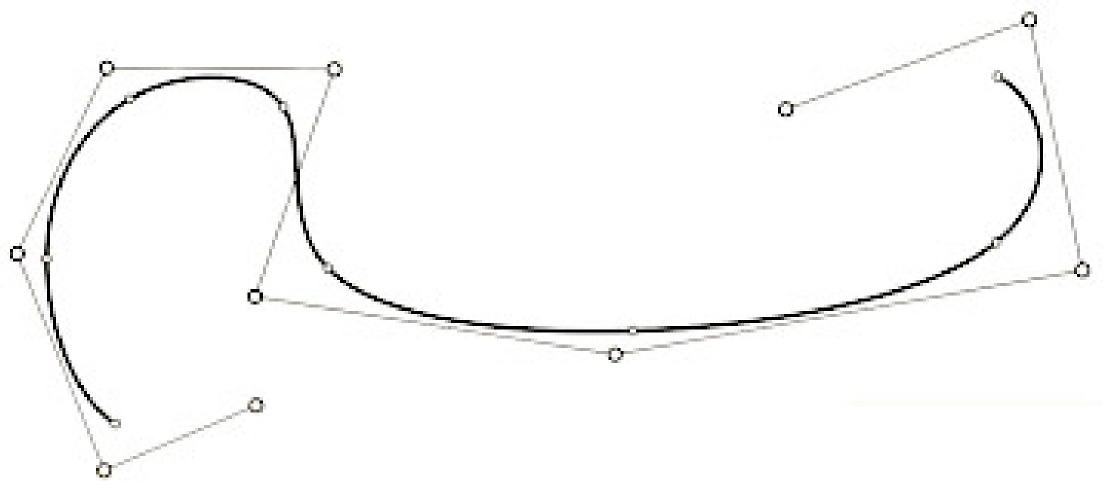
Tasks

$$P(t) = \begin{bmatrix} 1 & t & t^2 & t^3 \end{bmatrix} \begin{bmatrix} c_1 & c_2 & c_3 & c_4 \\ c_5 & c_6 & c_7 & c_8 \\ c_9 & c_{10} & c_{11} & c_{12} \\ c_{13} & c_{14} & c_{15} & c_{16} \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}$$

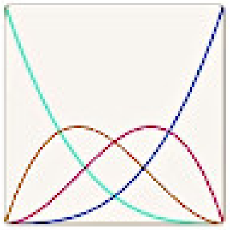
- compute $[c_i]$ to make $P(t)$ to be C^2 continuous (i.e., also G^2 continuous)

Solution

- C^2 implies C^1 and C^0 continuity between any two out of four basis functions $\rightarrow 3 \times \binom{4}{2} = 12$ constraints
- C^0, C^1 and C^2 continuity at the start (3 more constraints)
- the four basis functions must sum to 1 for $\forall t$ (the 16-th constraint) $\rightarrow$ weights or contributions of four control points P_0, P_1, P_2 and P_3

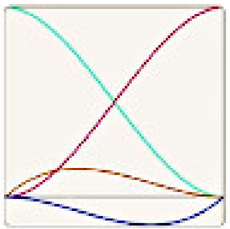


B-SPLINES



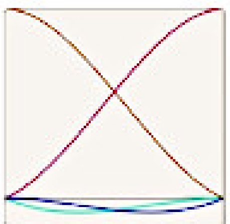
Bézier

$$\mathbf{P}(t) = \begin{bmatrix}1 & t & t^2 & t^3\end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{P}_0 \\ \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \end{bmatrix}$$



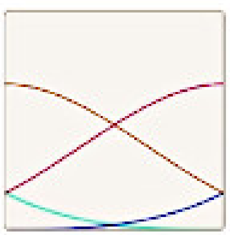
Hermite

$$\mathbf{P}(t) = \begin{bmatrix}1 & t & t^2 & t^3\end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & -2 & 3 & -1 \\ 2 & 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{P}_0 \\ \mathbf{v}_0 \\ \mathbf{P}_1 \\ \mathbf{v}_1 \end{bmatrix}$$



Catmull-Rom

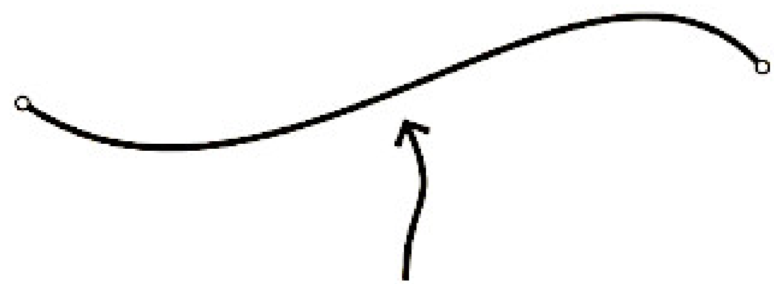
$$\mathbf{P}(t) = \begin{bmatrix}1 & t & t^2 & t^3\end{bmatrix} \frac{1}{2} \begin{bmatrix} 0 & 2 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 2 & -5 & 4 & -1 \\ -1 & 3 & -3 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{P}_0 \\ \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \end{bmatrix}$$



B-Spline

$$\mathbf{P}(t) = \begin{bmatrix}1 & t & t^2 & t^3\end{bmatrix} \frac{1}{6} \begin{bmatrix} 1 & 4 & 1 & 0 \\ -3 & 0 & 3 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{P}_0 \\ \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \end{bmatrix}$$

- **Uniform:** $0 \leq t \leq 1$
- **Cubic:** $\mathbf{P}(t) = \mathbf{a}t^3 + \mathbf{b}t^2 + \mathbf{c}t + \mathbf{d}$



This is not a spline, it's a cubic curve, <u>generated</u> by a spline

The Continuity of Splines

Freya Holmér
261K subscribers

YouTube HK

TAKE-HOME MESSAGES

Curves

- splines are curve generating procedures (via control points)
- non-uniform splines possible by adjusting knot distances
- bases in B-splines can be further scaled (by constants)

What matters

- numerical complexity and stability
- local control
- smoothness (parametric and geometric continuity)
- invariance to transformations and projections

Defining curves

- explicit mathematical expression (general and special polynomials)
- parametric expression: $P(t) = [x(t), y(t), z(t)] \in \mathcal{R}^3$
- implicit function: $f(x, y, z) = 0$
- projection into a plane
- constructive procedures
  $\rightarrow$ rolling a point on circle over a curve, Euclidean construction
- programmatic construction using language grammars

Take-Home Messages (cont.)

Generalizations

- parameter vector: $P(\boldsymbol{T}) = [x(\boldsymbol{T}), y(\boldsymbol{T}), z(\boldsymbol{T})] \in \mathcal{R}^3$, $\boldsymbol{T} \in \mathcal{R}^n$
- increase resolution by adding more control points
- define surfaces and manifold in $\mathcal{R}^n$ using rank-1 curves
- study intersections of curves and surfaces

Curves considered here

	Deg.	Cont.	Tangents	Interpol.	Use cases
Bézier	3	C^0/C^1	manual	some	shapes, fonts & vector graphics
Hermite	3	C^0/C^1	explicit	all	animation, physics sim & interpolation
Catmull-Rom	3	C^1	auto	all	animation & path smoothing
B-Spline	3	C^2	auto	none	curvature-sensitive shapes & animations, such as camera paths
Linear	1	C^0	auto	all	dense data & interpolation where smoothness doesn't matter

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Thank you!

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Recommended Resources

Geometric algebra (easy to follow introductions)
https://www.youtube.com/@sudgylacmoe/playlists

Vector calculus (and many other useful math concepts)
https://www.youtube.com/@Eigensteve/playlists

Splines (and other topics related to computer graphics)
https://www.youtube.com/@acegikmo/playlists

Famous curves (specific types)
https://mathshistory.st-andrews.ac.uk/Curves/

General mathematics (variety of topics)
https://mathworld.wolfram.com/
https://www.wikipedia.org/

Algebraic Concepts (selected applied math topics for SP/ML)
https://www.iaria.org/conferences2023/filesSIGNAL23/PavelLoskot_Keynote_AlgebraicConcepts.pdf