Numerical Experiments of Sensitivity Analysis for Fuzzy Reciprocal Matrix in AHP

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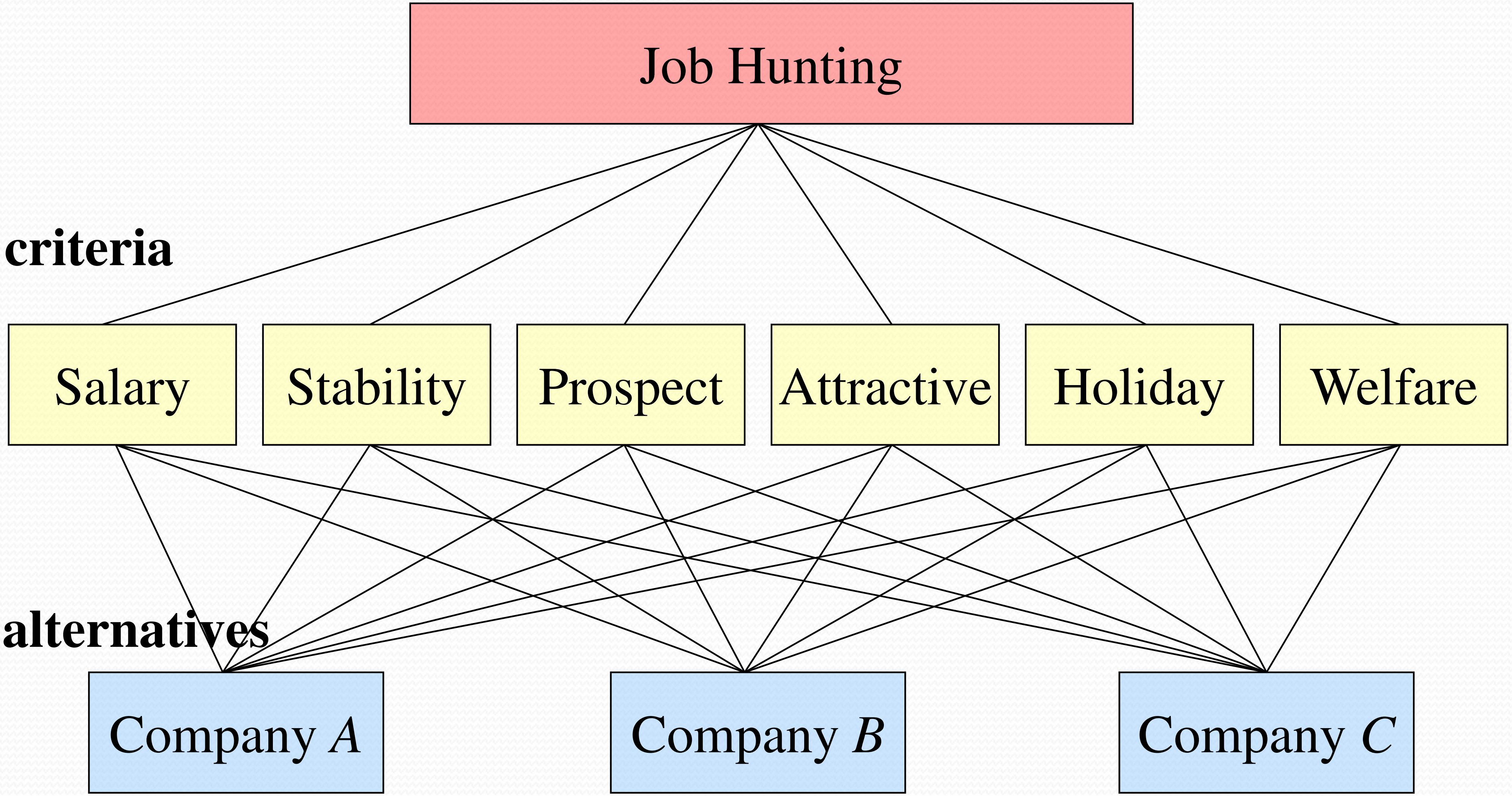
Introduction

- Analytic Hierarchy Process (AHP, Saaty 1977) has been a popular method in decision making
- It is difficult to keep reliability of data because of worsening of consistency index of crisp, non fuzzy, matrix (data in AHP)
- Fuzzy data AHP can prevent losing reliability, because it can reflect vagueness of decision maker's answers
- We propose and consider about a sensitivity analysis to investigate most influential components of fuzzy reciprocal data matrix through numerical experiments

Hierarchical structure in AHP

1. **Representation by a hierarchy**
2. Pairwise comparison matrices
3. (Consistency check)
4. Local weights of criteria
5. Global weights of alternative



Job Hunting

criteria

Salary

Stability

Prospect

Attractive

Holiday

Welfare

alternatives

Company *A*

Company *B*

Company *C*

Example

weights are normalized eigenvector corresponding to maximum eigenvalue

(P2)pairwise comparison matrix

$$\begin{array}{l} \\ \text{Salary} \\ \text{Stability} \\ \text{Prospect} \\ \text{Attractive} \\ \text{Holiday} \\ \text{Welfare} \end{array} \begin{array}{cccccc} \text{Salary} & \text{Stability} & \text{Prospect} & \text{Attractive} & \text{Holiday} & \text{Welf} \\ \end{array}$$

$$\begin{array}{l|cccccc} & \text{Salary} & \text{Stability} & \text{Prospect} & \text{Attractive} & \text{Holiday} & \text{Welf} \\ \hline \text{Salary} & 1 & 1/5 & 1/5 & 1/5 & 1/2 & 1/3 \\ \text{Stability} & & 1 & 3 & 4 & 7 & 5 \\ \text{Prospect} & & & 1 & 3 & 6 & 5 \\ \text{Attractive} & & & & 1 & 7 & 3 \\ \text{Holiday} & & & & & 1 & 1/5 \\ \text{Welfare} & & & & & & 1 \end{array}$$

- reciprocal data matrix
- **checking consistency**

C.I.=0.13

(P4)weights of criteria

$$\begin{pmatrix} \text{Salary} \\ \text{Stability} \\ \text{Prospect} \\ \text{Attractive} \\ \text{Holiday} \\ \text{Welfare} \end{pmatrix} = \begin{pmatrix} 0.04 \\ 0.41 \\ 0.26 \\ 0.16 \\ 0.04 \\ 0.09 \end{pmatrix}$$

(P4)weights of alternatives with respect to activities

	Company A	Company B	Company C
Salary	0.158	0.766	0.076
Stability	0.121	0.273	0.606
Prospect	0.180	0.778	0.042
Attractive	0.070	0.751	0.178
Holiday	0.157	0.249	0.594
Welfare	0.121	0.115	0.764

(P5)total weights of alternatives

$$\begin{pmatrix} \text{Company} & A \\ \text{Company} & B \\ \text{Company} & C \end{pmatrix} = \begin{pmatrix} 0.10 \\ 0.45 \\ 0.45 \end{pmatrix}$$

Consistency index of the pairwise comparison matrix *A* (checking reliability of data, C.I.)

$$\boxed{\mathrm{C.I.} = \frac{\lambda_A - n}{n - 1}}$$

where

A : comparison matrix with order n

λ_A : maximum eigenvalue (Frobnius root) of A

C.I. > 0.1, ⇒bad consistency

→ re-evaluate again **using sensitivity analysis**

Sensitivity analysis of eigenvalue for consistency

$$A(\varepsilon) = A + \varepsilon D_A$$

$A = a_{ij} \left(i, j = 1, \dots, n \right)$

perturbation $D_A = \left(a_{ij} d_{ij} \right)$

$\lambda(\varepsilon)$: eigenvalue of $A(\varepsilon)$

$$\boxed{\lambda(\varepsilon) = \lambda_A + \varepsilon \lambda^{(1)} + o(\varepsilon)}$$

eigenvalue of perturbed matrix

eigenvalue of original matrix

fluctuation

$$\lambda^{(1)} = \frac{1}{\boldsymbol{w}'_1 \boldsymbol{w}_2} \sum_{i}^{n} \sum_{j}^{n} w_{2i} a_{ij} w_{1j} d_{ij}$$

$\boldsymbol{w}_1 = (w_{1i})$: eigenvalue of A

$\boldsymbol{w}_2 = (w_{2i})$: eigenvalue of A'

Components of fuzzy data matrix

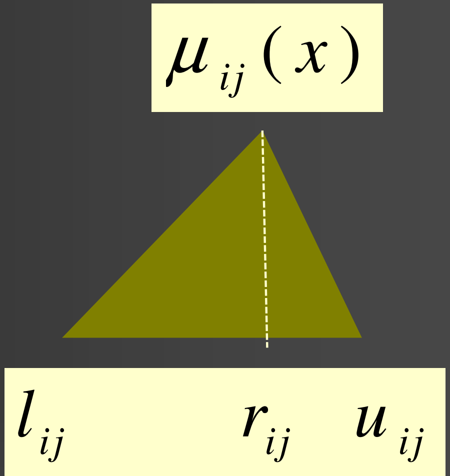
（Ohnishi, Dubois, Prade 2006）

fuzzy data

$$\tilde{r}_{ij} = (l_{ij}, r_{ij}, u_{ij})_\Delta$$

$$\mu_{ij}(r_{ij}) = 1$$

$$\mu_{ij}(l_{ij}) = \mu_{ij}(u_{ij}) = 0$$



reciprocity

$$\boxed{\mu_{ij}(r) = \mu_{ji}(1/r)} \Rightarrow$$

$$\mathrm{core}(\tilde{r}_{ji}) = 1/r_{ij}$$

$$\mathrm{supp}(\tilde{r}_{ji}) = [1/u_{ij}, 1/l_{ij}]$$

Optimal degree of satisfaction and weight of fuzzy data AHP

$$\boxed{\alpha^* \equiv \max_{w_1,\ldots,w_n} \min_{i,j} \left\{ \mu_{ij}\left(\frac{w_i}{w_j} \right) \right\}}$$

If all $\tilde{r}_{ij}$ $(i < j)$ are triangular fuzzy numbers $(l_{ij}, r_{ij}, u_{ij})_\Delta$,

[NLP]

Maximize α

$$w_j\{l_{ij} + \alpha(r_{ij} - l_{ij})\} \le w_i \le w_j\{u_{ij} + \alpha(r_{ij} - u_{ij})\}$$

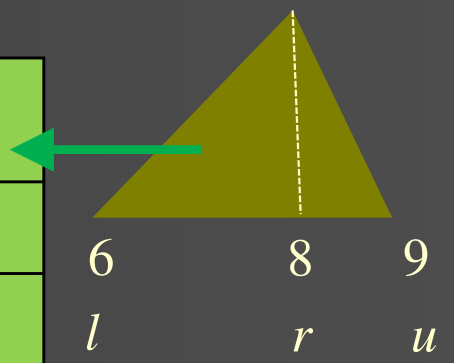
$$\sum_i^n w_i = 1 \qquad (i, j = 1,\ldots,n)$$

☆ ☆

example

fuzzy reciprocal data matrix

1	$(1, 3, 5)_\triangle$	$(2, 5, 7)_\triangle$	$(6, 8, 9)_\triangle$
	1	$(1, 2, 4)_\triangle$	$(2, 4, 5)_\triangle$
		1	$(0.5, 2, 3)_\triangle$
			1



α-cut interval matrix

1	[2.42,3.58]	[4.13,5.58]	[7.42,8.29]
	1	[1.71,2.58]	[3.42,4.29]
		1	[1.57,2.29]
			1

$[L_{ij},\ U_{ij}]$

$\alpha^*=0.711$

crisp matrix

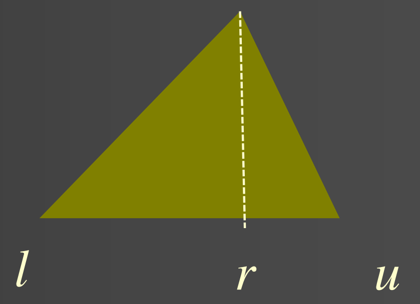
1	2.42	5.28	8.29
	1	2.18	3.42
		1	1.57
			1

w_1	0.581
w_2	0.240
w_3	0.110
w_4	0.070

Choice of crisp value for sensitivity analysis of consistency on fuzzy data

fuzzy data

$$\tilde{r}_{ij} = (l_{ij}, r_{ij}, u_{ij})_{\Delta}$$



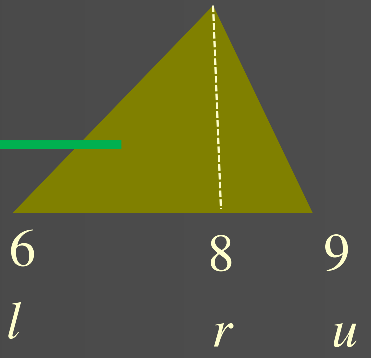
(1) core r of each component of matrix
(2) support set (interval $[l, u]$) of each component
(3) α-cut-set (interval $[L, U]$) of each component

- ✓ selection lower or upper value of intervals
  - ➢ only lower's, or upper's
  - ➢ all combination of the endpoints
- ✓ an endpoint of α-cut-set for calculating crisp weight must be meaningful

Numerical experiment: Sensitivity analysis of consistency on fuzzy data matrix

fuzzy reciprocal data matrix

1	$(1, 3, 5)_\triangle$	$(2, 5, 7)_\triangle$	$(6, 8, 9)_\triangle$
	1	$(1, 2, 4)_\triangle$	$(2, 4, 5)_\triangle$
		1	$(0.5, 2, 3)_\triangle$
			1



crisp matrix for analysis

$$A_N = \begin{pmatrix} 1 & 1 & 5.28 & 9 \\ & 1 & 2.18 & 2 \\ & & 1 & 0.5 \\ & & & 1 \end{pmatrix}$$

$$\text{C.I.}_N = \frac{\lambda_l - n}{n-1} = 0.108 > 0.1$$

- endpoint of side of α-cut-set for calculating weights

result of sensitivity analysis

	-0.050	0.035	**0.057**
		-0.030	**-0.020**
			-0.037

the biggest absolute value has most influence

Summary

Sensitivity analysis of consistency for fuzzy data AHP

- **Proposal and consideration about consistency on fuzzy pair-wise comparison matrix (reliability of data) by use of sensitivity analysis.**
- **As a choice of crisp value for sensitivity analysis**

✓ Selection of an endpoint of α-cut-set for calculating crisp weight must be more meaningful than using other value.

In the future

- **Other indices for consistency**
- **More experiments using real data**