An Evaluation of Reliability on Pair-wise Comparison Matrix in Fuzzy Data AHP

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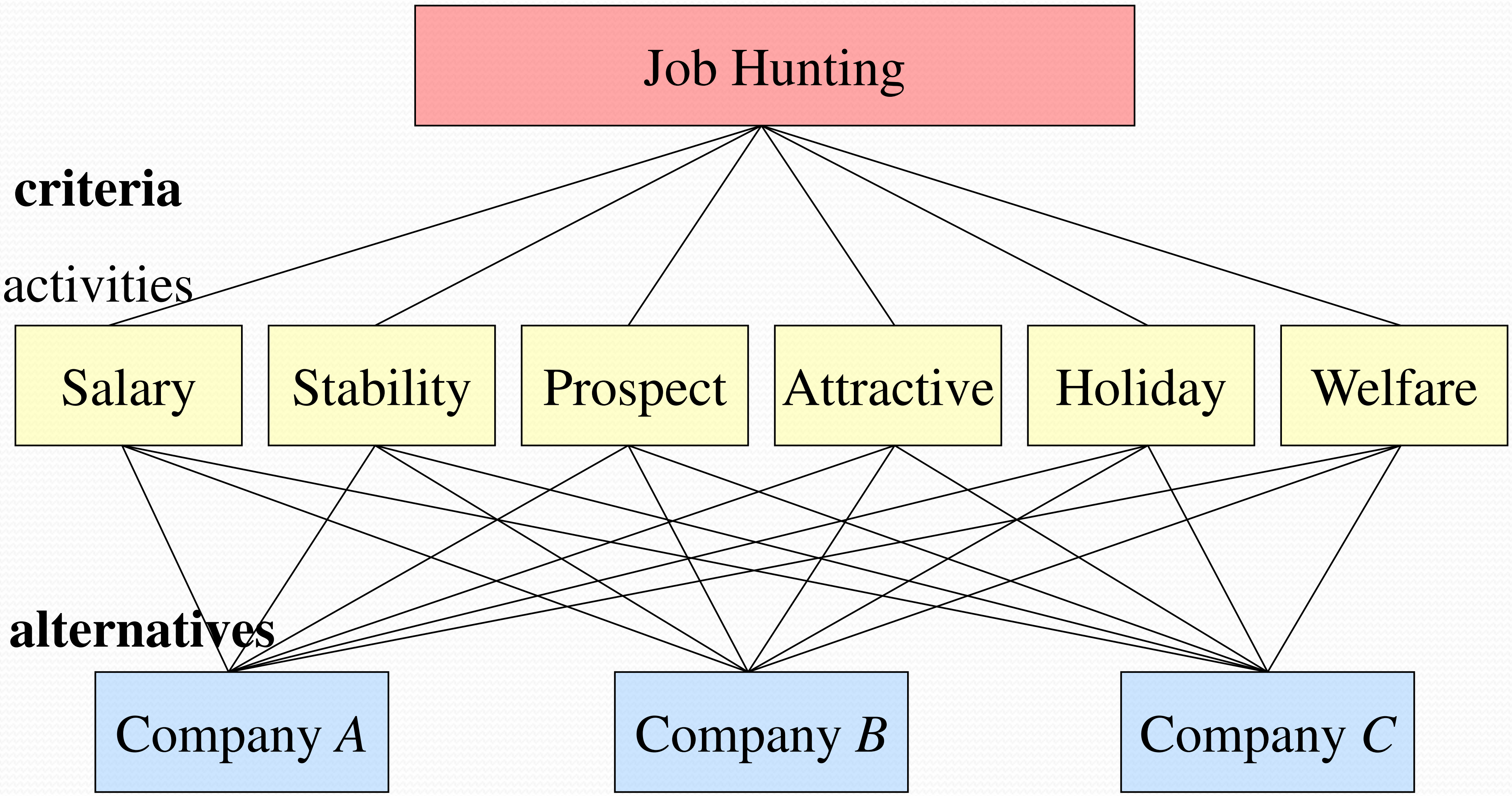
Introduction

- Analytic Hierarchy Process (AHP) has been widely used in decision making. However, it is difficult to keep reliability of data because of worsening of consistency index of pair-wise comparison (data) matrix.
- For this situation, fuzzy data AHP is useful as an extension method.
- In this research, we consider an evaluation of consistency index on fuzzy data reciprocal matrix by use of a sensitivity analysis.

Hierarchy structure

1. **Representation by a hierarchy**
2. Pairwise comparison matrices
3. (Consistency check)
4. Local weights of criteria
5. Global weights of alternative



Pairwise comparison matrix (data)

table1.3 Definition Intensity of importance

Intensity of importance	Definition
1	Equal importance
3	Moderate importance
5	Strong importance
7	Very strong importance
9	Extreme importance

Even number is used if priority is between elements

Reciprocal is used without priority

table1.4 pairwise comparison matrix

	A_1	A_2	A_3	weight
A_1	1	3	2	0.508
A_2	1/3	1	1/5	0.113
A_3	1/2	5	1	0.379

$$X = (x_{ij})$$

$$x_{ii} = 1,\ x_{ij} \cdot x_{ji} = 1\ \text{(reciplocal)}$$

$$i, j = 1,\ \ldots,\ n$$

- **Weights are normalized eigenvector** corresponding to the maximum eigenvalue.
- **Checking consistency is necessary .**

Consistency index(C.I.)

cheking reliability

- The consistency of a comparison matrix is usually measured by the consistency index C.I.

$$\text{C.I.} = \frac{\lambda - n}{n-1}$$

λ : **maximum eigenvalue**

n : size of matrix

C.I. ≧ 0.1, ⇒bad consistency

→ re-evaluate again

Sensitivity analysis of consistency

$$A(\varepsilon) = A + \varepsilon D_A \qquad \lambda(\varepsilon) \text{ : eigenvalue of } A(\varepsilon)$$

$A = a_{ij}\ (i, j = 1, \ldots, n)$

perturbation $D_A = (a_{ij} d_{ij})$

$$\lambda(\varepsilon) = \lambda_A + \varepsilon \lambda^{(1)} + o(\varepsilon)$$

Eigenvalue of perturbed matrix

Eigenvalue of not perturbed matrix

Fluctuation

$$\lambda^{(1)} = \frac{1}{\mathbf{w'}_1 \mathbf{w}_2} \sum_{i}^{n} \sum_{j}^{n} w_{2i} a_{ij} w_{1j} d_{ij}$$

Example of sensitivity analysis for consistency

table2.1 pairwise comparison matrix

	A	B	C	D	weight
A	1	2	5	7	0.489
B	1/2	1	4	7	0.326
C	1/5	1/4	1	8	0.146
D	1/7	1/3	1/8	1	0.040

C.I.=0.132

table2.2 example of sensitivity analysis of consistency

	A	B	C	D
A		0.149	0.184	-0.332
B			0.265	-0.117
C				0.449
D				

The biggest absolute value has most influence.

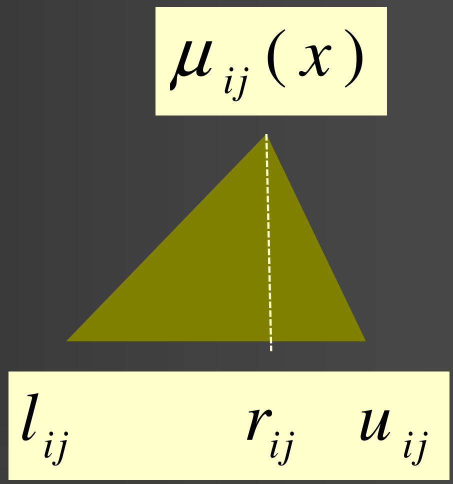
$$C.I.^{(1)} = \sum_{i}^{n}\sum_{j}^{n} g_{ij} d_{ij}$$

Componets of fuzzy data matrix

(Ohnishi, Dubois, Prade 2006)

fuzzy data

$$\tilde{r}_{ij} = (l_{ij}, r_{ij}, u_{ij})_{\Delta}$$



$$\mu_{ij}(r_{ij}) = 1$$

$$\mu_{ij}(l_{ij}) = \mu_{ij}(u_{ij}) = 0$$

Reciprocity

$$\mu_{ij}(r) = \mu_{ji}(1/r)$$

$$\text{core}(\tilde{r}_{ji}) = 1/r_{ij}$$

$$\text{supp}(\tilde{r}_{ji}) = [1/u_{ij}, 1/l_{ij}]$$

Optimal degree of satisfaction and weight

$$\alpha^{*} \equiv \max_{w_1,\ldots,w_n} \min_{i,j} \left\{ \mu_{ij}\left(\frac{w_i}{w_j} \right) \right\}$$

If all $\tilde{r}_{ij}$ $(i < j)$ are triangular fuzzy numbers $(l_{ij}, r_{ij}, u_{ij})_{\Delta}$,

[NLP]

Maximize α

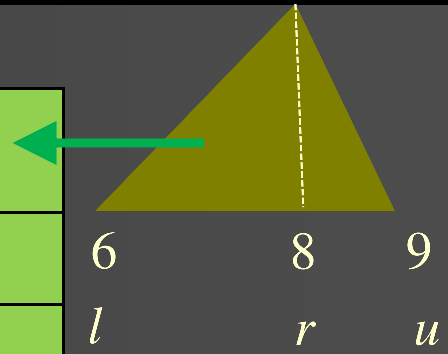
$$w_j\{l_{ij} + \alpha(r_{ij} - l_{ij})\} \le w_i \le w_j\{u_{ij} + \alpha(r_{ij} - u_{ij})\}$$

$$\sum_{i}^{n} w_i = 1 \qquad (i, j = 1,\ldots,n)$$

☆ ☆

fuzzy data

1	(1, 3, 5)△	(2, 5, 7)△	(6, 8, 9)△
	1	(1, 2, 4)△	(2, 4, 5)△
		1	(0.5, 2, 3)△
			1



α-cut interval data

1	[2.42,3.58]	[4.13,5.58]	[7.42,8.29]
	1	[1.71,2.58]	[3.42,4.29]
		1	[1.57,2.29]
			1

$[L_{ij}, U_{ij}]$

$\alpha^{*}=0.711$

crisp data

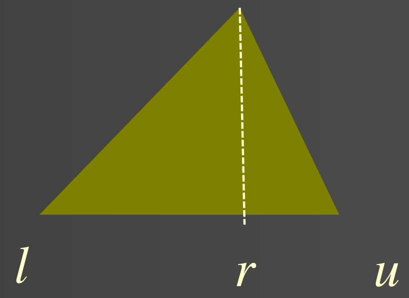
1	2.42	5.28	8.29
	1	2.18	3.42
		1	1.57
			1

w_1	0.581
w_2	0.240
w_3	0.110
w_4	0.070

Sensitivity analysis of consistency index on fuzzy data pair-wise comparison matrix

Fuzzy number

$$\tilde{r}_{ij} = (l_{ij}, r_{ij}, u_{ij})_\Delta$$



(1)analysis on core r

(2)analysis on support set（interval[l,u]）

- ✓ Combination of endpoints l,u
  - ➢ all combination
  - ➢ endpoint of α-cut-set for calculating weights
  - ➢ endpoint of Core greater than 1

(3) analysis on α-cut-set (interval [L,U])

Conclusions

- ✓ Fuzzy number data AHP
- ✓ Sensitivity analysis for consistency
- ◆ Consider an evaluation of consistency index on fuzzy data pair-wise comparison matrix, by use of sensitivity analysis.
- ◆ Selecting core, endpoints of fuzzy data or α-cut set.

In the future

- ◆ Other index for consistency
- ◆ Numerical experiment using real data.